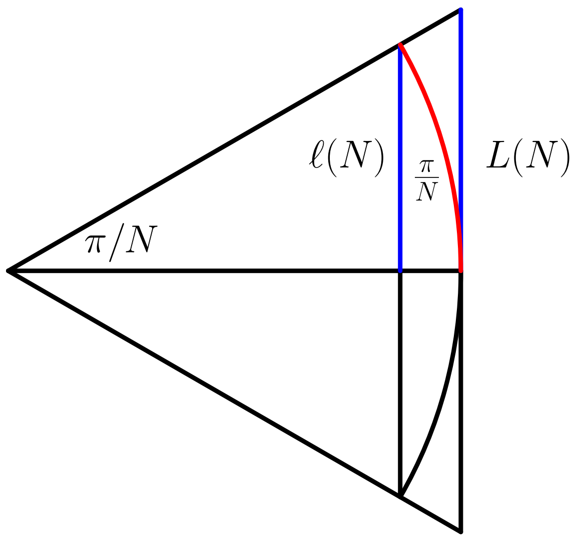
Approximation of π

Hiroyuki Chihara

Recall that the perimeter of the hemicircle with radius 1 is π. Consider the internal regular N-gon and the external regilar N-gon of the circle with radious 1. Denote by $\ell(N)$ and $L(N)$ the half of the perimeter of them respectively.



Compare the length $2\sin(\pi/N)$ of the vertical edge of the figure of the isosceles triangle of the internal regular N-gon and the length $2\pi/N$ of the arc of the sector, and compare the area π/N of the sector and the are $\tan(\pi/N)$ of the isosceles triangle of the external regular N-gon. Then we have

$$\ell(N) = N \cdot \sin\left(\frac{\pi}{N}\right) \leqq \pi \leqq L(N) = N \cdot \tan\left(\frac{\pi}{N}\right),$$

$$L(N), \ell(N) \to \pi \quad (N \to \infty).$$

We begin with regular N_0-gon with some $N_0 = 3, 4, 5, \dots$, and compute $\ell(N_0 \cdot 2^n)$ and $L(N_0 \cdot 2^n)$. If $\cos\bigl(\pi/(N_0 \cdot 2^n)\bigr)$ is given, then we have

$$\ell(N_0 \cdot 2^n) = 2^n N_0 \sqrt{1 - \cos^2\left(\frac{\pi}{N_0 \cdot 2^n}\right)},$$

$$L(N_0 \cdot 2^n) = \frac{\ell(N_0 \cdot 2^n)}{\cos\left(\frac{\pi}{N_0 \cdot 2^n}\right)},$$

$$\cos\left(\frac{\pi}{N_0 \cdot 2^{n+1}}\right) = \sqrt{\frac{1 + \cos\left(\frac{\pi}{N_0 \cdot 2^n}\right)}{2}}.$$

We compute the half perimeter of the regular 3×2^n-gon and the regular 4×2^n-gon by using the LibreOffice calc or the free computer algebra system Maxima below. LibreOffice calc cannot proceed at some step due to the accumulation of numerical errors coming from the induction. Maxima computes each step independently by using trigonometric functions.

Compuation of the half of perimeter of internal and external regular polygons of the circle with radius 1

regular 3*2^n-gons			
n	cos(pi/2^n*3)	l(2^n*3)	L(2^n*3)
0	0.5	2.59807621135332	5.19615242270663
1	0.866025403784439	3	3.46410161513775
2	0.965925826289068	3.10582854123026	3.21539030917348
3	0.99144486137381	3.1326286132813	3.15965994209757
4	0.997858923238603	3.13935020304722	3.14608621513179
5	0.999464587476366	3.14103195088906	3.14271459964392
6	0.999866137909562	3.14145247228274	3.1418730499771
7	0.999966533917401	3.14155760791292	3.14166274705792
8	0.999991633444351	3.14158389207598	3.14161017653235
9	0.9999979083589	3.14159046336183	3.1415970344553
10	0.999999477089588	3.14159210687681	3.14159374964889
11	0.999999869272389	3.14159251058506	3.141592921278
12	0.999999967318097	3.14159259729645	3.14159269996968
13	0.999999991829524	3.14159265599339	3.14159268166169
14	0.999999997957381	3.14159264532122	3.14159265544129
15	0.999999999489345	3.14159264532122	3.1415926602541
∞	1	3.14159265358979	3.14159265358979
regular 4*2^n-gons			
n	cos(pi/2^n*4)	l(2^n*4)	L(2^n*4)
0	0.707106782373095	2.82842712	2.12132034
1	0.923879532832364	3.06146745271952	3.31370849112102
2	0.980785280485072	3.12144514567492	3.18259787109685
3	0.995184726692756	3.13654848386625	3.15172490065214
4	0.998795456210318	3.14033115025125	3.14411837851812
5	0.999698818697491	3.1412772442226	3.14222362322622
6	0.999924701839466	3.14151379443855	3.1417503624617
7	0.999981175282681	3.14157293370005	3.14163207403576
8	0.999995293809596	3.14158771862837	3.14160250360793
9	0.999998823451707	3.14159141461219	3.14159511085055
10	0.999999705862883	3.14159234086789	3.14159326492707
11	0.999999926465718	3.14159257061602	3.14159280163079
12	0.999999981616429	3.1415926500644	3.14159270781809
13	0.999999995404107	3.14159272121221	3.14159273565063
14	0.999999998851027	3.14159230381174	3.14159230742134
∞	1	3.14159265358979	3.14159265358979

(%i1)
```
/* interlal regular 3*2^n-gons
   external regular 3*2^n-gons
   n=0,1,2,...
   The computation of the half perimeter approximating π */
for n : 0 thru 26 do
print(n, float(3*2^n*sin(%pi/(3*2^n))), float(3*2^n*tan(%pi/(3*2^n))));
```

```
0 2.598076211353315 5.196152422706631
1 3.0 3.464101615137754
2 3.105828541230249 3.215390309173472
3 3.132628613281237 3.159659942097500
4 3.139350203046866 3.146086215131435
5 3.141031950890509 3.142714599645368
6 3.141452472285461 3.141873049979823
7 3.141557607911857 3.141662747056848
8 3.141583892148317 3.141610176604689
9 3.141590463228049 3.141597034321525
10 3.141592105999271 3.141593748771352
11 3.141592516692156 3.141592927385097
12 3.141592619365383 3.141592722038613
13 3.14159264503369 3.141592670701998
14 3.141592651450767 3.141592657867844
15 3.141592653055037 3.141592654659305
16 3.141592653456103 3.141592653857171
17 3.14159265355637 3.141592653656637
18 3.141592653581437 3.141592653606504
19 3.141592653587704 3.141592653593971
20 3.141592653589271 3.141592653590837
21 3.141592653589662 3.141592653590054
22 3.14159265358976 3.141592653589858
23 3.141592653589785 3.141592653589809
24 3.141592653589791 3.141592653589796
25 3.141592653589792 3.141592653589794
26 3.141592653589793 3.141592653589793
```

(%o1) *done*

(%i2)
```
float(%pi);
```

(%o2) 3.141592653589793

(%i3)
```
/* interlal regular 4*2^n-gons
   external regular 4*2^n-gons
   n=0,1,2,...
   The computation of the half perimeter approximating π */
for n : 0 thru 26 do
print(n, float(2^(n+2)*sin(%pi/2^(n+2))), float(2^(n+2)*tan(%pi/2^(n+2))));
```

```
0 2.82842712474619 4.0
1 3.061467458920718 3.31370849898476
2 3.121445152258052 3.182597878074528
3 3.136548490545939 3.151724907429256
4 3.140331156954753 3.144118385245904
5 3.141277250932773 3.14223629942457
6 3.141513801144301 3.141750369168966
7 3.141572940367091 3.141632080703182
8 3.141587725277159 3.141602510256809
9 3.141591421511199 3.141595117749589
10 3.141592345570117 3.141593269629307
11 3.14159257658487 2 3.141592807599644
12 3.141592634338562 3.141592692092254
13 3.141592648776985 3.141592663215408
14 3.141592652386591 3.141592655996197
15 3.141592653288993 3.141592654191394
16 3.141592653514593 3.141592653740193
17 3.141592653570993 3.141592653627393
18 3.141592653585093 3.141592653599193
19 3.141592653588618 3.141592653592143
20 3.141592653589499 3.14159265359038
21 3.14159265358972 3.14159265358994
22 3.141592653589775 3.14159265358983
23 3.141592653589788 3.141592653589802
24 3.141592653589792 3.141592653589795
25 3.141592653589792 3.141592653589793
26 3.141592653589793 3.141592653589793
```

(%o3) *done*

(%i4) `float(%pi);`

(%o4) 3.141592653589793