Isoperimetric inequality and the area of isoperimetric regular polygons

Hiroyuki Chihara

Let γ be a piecewise smooth simple closed plane curve, that is, a closed plane curve consisting of finitely many C^1-curves without self-crossing, and let Ω be the bounded domain enclosed by γ. Denote the length of γ and the area of Ω by $L(\gamma)$ and $A(\Omega)$ respectively.

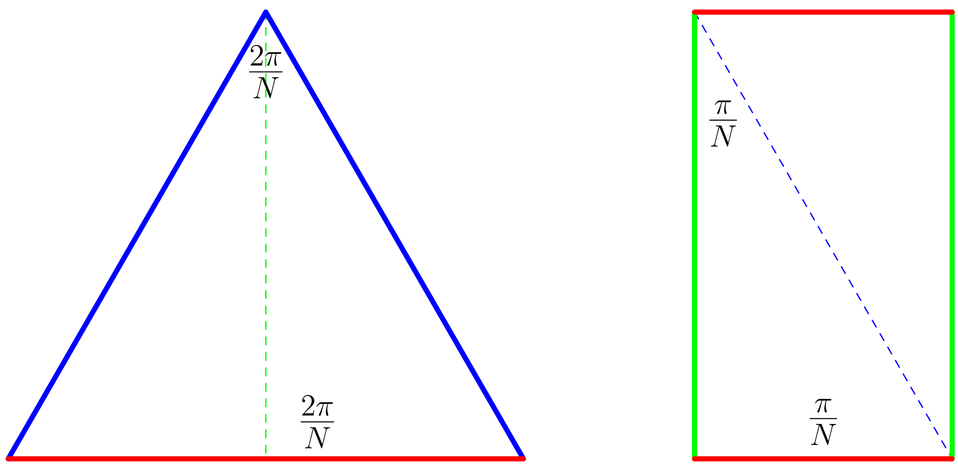
Isoperimetric Problem
Fix the length of simple plane curves. Find the shape of the curve enclosing the maximum area.

The answer is the following.

定理 1. *$A(\Omega) \leqq L(\gamma)^2/4\pi$ holds. The identity holds if and only if the curve is the circle.*

In what follows we compute the area $S(N)$ of regular N-gons with the perimeter $L(\gamma) = 2\pi$, which is the same as the perimeter of the circle with radius 1. Of course $S(N) \leqq \pi$ theoretically.

$$\begin{aligned}
S(N) &= N \times \text{the area of isosceles triangle with the bottom length } 2\pi/N \text{ and the apex angle } 2\pi/N \\
&= N \times \frac{\pi}{N} \times \frac{\cos(\pi/N)}{\sin(\pi/N)} \cdot \frac{\pi}{N} \\
&= \frac{\pi^2}{N} \cdot \frac{\cos(\pi/N)}{\sin(\pi/N)} \\
&= \pi \cdot \cos(\pi/N) \cdot \frac{\pi/N}{\sin(\pi/N)} \to \pi \cdot 1 \cdot 1 = \pi \quad (N \to \infty).
\end{aligned}$$



Let N_0 be an integer greater than or equal to 3. We compute $S(N_0{\cdot}2^n)$ inductively for $n = 0, 1, 2, \ldots$. If $\cos\bigl(\pi/(N_0{\cdot}2^n)\bigr)$ is given with some n, then

$$S(N_0{\cdot}2^n N_0) = \frac{\pi^2}{N_0 \cdot 2^n} \cdot \frac{\cos\bigl(\pi/(N_0 \cdot 2^n)\bigr)}{\sqrt{1 - \cos\bigl(\pi/(N_0 \cdot 2^n)\bigr)^2}},$$

$$\cos\bigl(\pi/(N_0 \cdot 2^{n+1})\bigr) = \sqrt{\frac{1 + \cos\bigl(\pi/(N_0 \cdot 2^n)\bigr)}{2}}.$$

Inductive computation by the spreadsheet LibreOffice calc and direct computation by the free computer algebra system Maxima are the following.

The area of regular polygons

regular 3*2^n-gons		
n	cos(pi/(n*2^n))	S(3*2^n)
0	0.5	1.89940625258801
1	0.866025403784439	2.84910937888202
2	0.965925826289068	3.06948875628924
3	0.99144486137381	3.1236286758560 2
4	0.997858923238603	3.1371055102115
5	0.999464587476366	3.140471108068
6	0.999866137909562	3.14131228222384
7	0.999966533917401	3.14152256168552
8	0.999991633444351	3.14157513066159
9	0.9999979083589	3.14158827281376
10	0.999999477089588	3.14159155853159
11	0.999999869272389	3.14159238056552
12	0.999999967318097	3.14159258586556
13	0.999999991829524	3.14159264686223
14	0.999999997957381	3.14159265544129
∞	1	3.14159265358979
regular 4*2^n-gons		
n	cos(pi/(4*2^n))	S(4*2^n)
0	0.707106782373095	2.46740110855309
1	0.923879532832364	2.97841660711395
2	0.980785280485072	3.10111575537753
3	0.995184726692756	3.13149297993855
4	0.998795456210318	3.13906895762062
5	0.999698818697491	3.1409618106543
6	0.999924701839466	3.14143495263915
7	0.999981175282681	3.14155323367551
8	0.999995293809596	3.14158280367665
9	0.999998823451707	3.14159019618272
10	0.999999705862883	3.14159204403132
11	0.999999926465718	3.14159250792039
12	0.999999981616429	3.14159261833423
13	0.999999995404107	3.14159264741994
14	0.999999998851027	3.14159269619426
∞	1	3.14159265358979

→ /* The area of regular polygons with perimeter 2π */

```
(%i2) /* The area of regular 3·2^n-gons with perimeter 2π */
for n : 0 step 1 thru 27 do
print(n, float( %pi^2·cos(%pi/(3·2^n))/( 3·2^n·sin(%pi/(3·2^n)) )));
float(%pi);
0 1.899406252588019
1 2.849109378882028
2 3.06948875628924
3 3.12362867585603
4 3.137105510211528
5 3.140471108067868
6 3.141312282223732
7 3.141522561686589
8 3.141575130672635
9 3.141588272864169
10 3.141591558408616
11 3.141592379794513
12 3.141592585140973
13 3.141592636477588
14 3.141592649311742
15 3.14159265252028
16 3.141592653322415
17 3.141592653522949
18 3.141592653573082
19 3.141592653585615
20 3.141592653588748
21 3.141592653589531
22 3.141592653589728
23 3.141592653589776
24 3.141592653589788
25 3.141592653589792
26 3.141592653589792
27 3.141592653589793
(%o1) done
(%o2) 3.141592653589793
```

```
(%i4) /* The area of regular 4*2^n-gons with perimeter 2π */
for n : 0 step 1 thru 25 do
print(n, float( (%pi^2 * cos(%pi/2^(n+2)) )/( 2^(n+2) * sin(%pi/2^(n+2))) ));
float(%pi);
0 2.467401100272339
1 2.978416600045889
2 3.101115748578475
3 3.13149297320483
4 3.139068950903211
5 3.140961803940765
6 3.141434945927927
7 3.14155322697121
8 3.141582796953702
9 3.14159018943193
10 3.1415920375504
11 3.14159249957995
12 3.141592615087332
13 3.141592643964177
14 3.141592651183389
15 3.141592652988192
16 3.141592653439393
17 3.141592653552193
18 3.141592653580392
19 3.141592653587443
20 3.141592653589205
21 3.141592653589646
22 3.141592653589756
23 3.141592653589784
24 3.141592653589791
25 3.141592653589793
(%o3) done
(%o4) 3.141592653589793
```