

Microlocal analysis of d -plane transform on the Euclidean space

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1-1. d -plane transform

- $n = 2, 3, 4, \dots$, $d = 1, \dots, n - 1$.
- The Grassmannian $\mathbf{G}_{d,n}$ is the set of all d -dimensional vector subspaces of \mathbb{R}^n . $\dim \mathbf{G}_{d,n} = d(n - d)$.
- The affine Grassmannian $\mathbf{G}(d, n)$ is the set of all d -dimensional planes in \mathbb{R}^n , that is, $\mathbf{G}(d, n) = \{\mathbf{x}'' + \sigma : \sigma \in \mathbf{G}_{d,n}, \mathbf{x}'' \in \sigma^\perp\}$.
 $\mathbf{N}(d, n) := \dim \mathbf{G}(d, n) = (d + 1)(n - d)$.
 $\mathbf{x}'' + \sigma$ is sometimes denoted by (σ, \mathbf{x}'') .
- Denote $\mathbf{x} = \mathbf{x}' + \mathbf{x}'' \in \sigma \oplus \sigma^\perp = \mathbb{R}^n$. The d -plane transform of $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}' + \mathbf{x}'') = \mathcal{O}(\langle \mathbf{x} \rangle^{-d-\varepsilon})$ is defined by

$$\mathcal{R}_d \mathbf{f}(\sigma, \mathbf{x}'') = \int_{\sigma} \mathbf{f}(\mathbf{x}' + \mathbf{x}'') d\mathbf{x}', \quad (1)$$

where $\langle \mathbf{x} \rangle = \sqrt{\mathbf{1} + |\mathbf{x}|^2}$ and $d\mathbf{x}'$ is the Lebesgue measure on σ .

- $\mathcal{R}_1 \mathbf{f}$ is said to be the X-ray transform of \mathbf{f} ,
and $\mathcal{R}_{n-1} \mathbf{f}$ is said to be the Radon transform \mathbf{f} .

1-2. Filtered back-projection

The formal adjoint of \mathcal{R}_d is given by

$$\begin{aligned}\mathcal{R}_d^* \varphi(\mathbf{x}) &= \frac{1}{\mathbf{C}(\mathbf{d}, n)} \int_{\{\Xi \in \mathbf{G}(\mathbf{d}, n) : \mathbf{x} \in \Xi\}} \varphi(\Xi) d\mu(\Xi) \\ &= \frac{1}{\mathbf{C}(\mathbf{d}, n)} \int_{\mathbf{O}(n)} \varphi(\mathbf{x} + \mathbf{k} \cdot \sigma) d\mathbf{k},\end{aligned}$$

where $\mathbf{x} \in \mathbb{R}$, $\varphi \in \mathbf{C}(\mathbf{G}(\mathbf{d}, n))$,

$\mathbf{C}(\mathbf{d}, n) = (4\pi)^{d/2} \Gamma(n/2) / \Gamma((n-d)/2)$, $d\mu$ and $d\mathbf{k}$ are normalized measure, and $\sigma \in \mathbf{G}_{d,n}$.

Proposition 1 (FBP (filtered back-projection))

For $f(\mathbf{x}) = \mathcal{O}(\langle \mathbf{x} \rangle^{-d-\varepsilon})$,

$$f = (-\Delta_{\mathbf{x}})^{d/2} \mathcal{R}_d^* \mathcal{R}_d f = \mathcal{R}_d^* (-\Delta_{\mathbf{x}''})^{d/2} \mathcal{R}_d f, \quad (2)$$

where $-\Delta_{\mathbf{x}} = -\partial_{x_1}^2 - \dots - \partial_{x_n}^2$, and $-\Delta_{\mathbf{x}''}$ is the Laplacian on σ^\perp .

1-3. Range of the d -plane transform

Proposition 2

$\mathcal{R}_d : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}_H(\mathbf{G}(d, n))$ is bijective, where $\mathcal{D}_H(\mathbf{G}(d, n))$ is the set of all $\varphi \in \mathcal{D}(\mathbf{G}(d, n))$ with the following conditions: for any $k = 0, 1, 2, \dots$, there exists a homogeneous polynomial \mathbf{P}_k on \mathbb{R}^n of degree k such that for any $\sigma \in \mathbf{G}_{d,n}$,

$$\mathbf{P}_k(\zeta'') = \int_{\sigma^\perp} \varphi(\sigma, \mathbf{x}'') (\zeta'' \cdot \mathbf{x}'')^k d\mathbf{x}'', \quad \zeta'' \in \sigma^\perp.$$

Note that if $\mathbf{f}(\mathbf{x}) \in \mathcal{D}(\mathbb{R}^n)$, then we can define $\mathbf{P}_k(\zeta)$ by

$$\mathbf{P}_k(\zeta) := \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}) (\zeta \cdot \mathbf{x})^k d\mathbf{x}, \quad \zeta \in \mathbb{R}^n.$$

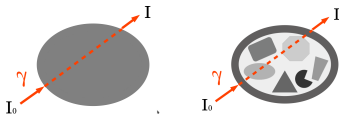
For any $\sigma \in \mathbf{G}_{d,n}$ and $\zeta'' \in \sigma^\perp$,

$$\mathbf{P}_k(\zeta'') = \int_{\sigma^\perp} \left(\int_{\sigma} \mathbf{f}(\mathbf{x}' + \mathbf{x}'') d\mathbf{x}' \right) (\zeta'' \cdot \mathbf{x}'')^k d\mathbf{x}''.$$

1-4. CT scanner

Consider the following situation on \mathbb{R}^2 :

- Here is an object whose attenuation coefficient distribution is $\mathbf{f}(\mathbf{x})$.
- The X-ray beam is supposed to have no width, and traverses the object along a line γ . I_0 and I denote the intensities of the beam before and after passing through the object respectively.

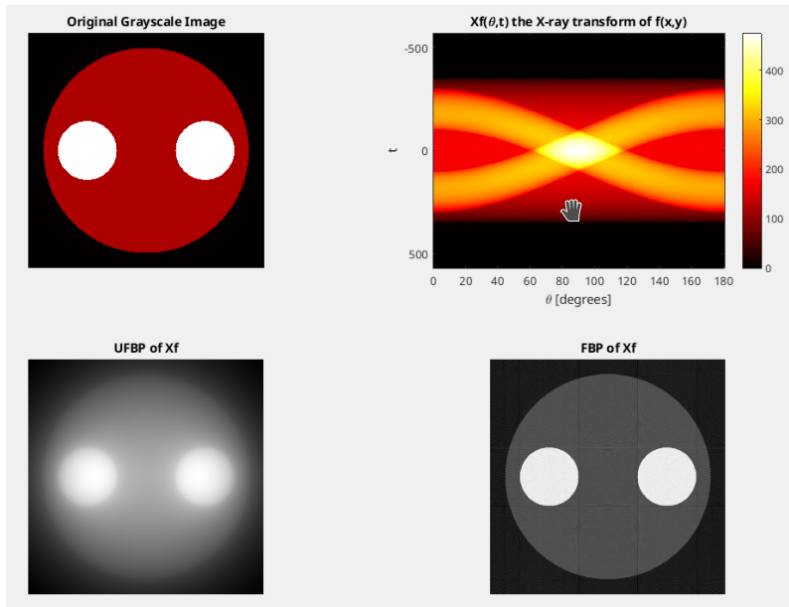


If the object is uniform, that is, $\mathbf{f} = \mathbf{d} \cdot \chi_{\Omega}$ and the travel length in the object is ℓ , then the Beer-Lambert law obtains

$$\log \left(\frac{I_0}{I} \right) = \mathbf{d} \cdot \ell = \int_{\gamma} \mathbf{f} = \mathcal{R}_1 \mathbf{f}(\gamma).$$

The same formula can be obtained for more general \mathbf{f} , and we can regard $\mathcal{R}_1 \mathbf{f}(\gamma)$ as the measurement of CT scanners.

1-5. MATLAB illustrates \mathcal{R}_1 , UFBP and FBP.



1-6. Beam hardening

- There are some factors causing artifacts in CT images: beam width, partial volume effect, beam hardening, noise in measurements, numerical errors and etc.
- In the formulation of CT scanners in Page 6, the X-ray is supposed to be monochromatic with a fixed energy, say $E_0 > 0$.
- Actually, however, the X-ray beam has a wide range of energy E and the attenuation coefficient distribution f_E depends on E . This is described by the spectral function $\rho(E)$ which is a probability density function of $E \in [0, \infty)$. The formulation of the measurements of CT scanners becomes

$$\log \left(\frac{I_0}{I} \right) = - \log \left\{ \int_0^\infty \rho(E) \exp(-\mathcal{R}_1 f_E) dE \right\}.$$

If f_E is independent of E , that is, $f_E = f_{E_0}$, then

$$\log \left(\frac{I_0}{I} \right) = - \log \left\{ \exp(-\mathcal{R}_1 f_{E_0}) \cdot \int_0^\infty \rho(E) dE \right\} = \mathcal{R}_1 f_{E_0}.$$

1-7. Metal streaking artifacts

- Consider a simple model of the beam hardening:

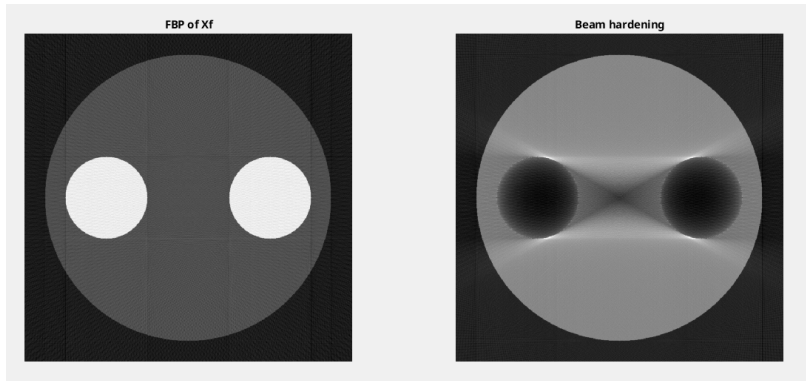
$$\begin{aligned}\rho(\mathbf{E}) &= \frac{1}{2\epsilon} \chi_{[E_0-\epsilon, E_0+\epsilon]}(\mathbf{E}), \\ \mathbf{f}_{\mathbf{E}}(\mathbf{x}) &= \mathbf{f}_{E_0}(\mathbf{x}) + \alpha(\mathbf{E} - \mathbf{E}_0)\chi_D(\mathbf{x}),\end{aligned}$$

where \mathbf{f}_{E_0} is an attenuation coefficient distribution of human tissue, ϵ and α are small positive constants, and D is a metal region. Then the measurement becomes

$$\begin{aligned}\mathbf{P} &= \mathcal{R}_1 \mathbf{f}_{E_0} - \log \left\{ \frac{\sinh(\alpha \epsilon \mathcal{R}_1 \chi_D)}{\alpha \epsilon \mathcal{R}_1 \chi_D} \right\} \\ &= \mathcal{R}_1 \mathbf{f}_{E_0} + \sum_{k=1}^{\infty} \mathbf{A}_k (\alpha \epsilon \mathcal{R}_1 \chi_D)^{2k}.\end{aligned}$$

- Park-Choi-Seo (CPAM, 2017) proved that the metal streaking artifacts occur and they are described in the wave front set.
- Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) proved that the streaking artifacts are conormal distributions.

1-8. MATLAB illustrates metal streaking artifacts.



- Left: $\mathcal{R}_1^*(-\Delta_{x''})(\mathcal{R}_1 f_{E_0} + \alpha \varepsilon \mathcal{R}_1 \chi_D)$.
- Right: $\mathcal{R}_1^*(-\Delta_{x''}) \left\{ \mathcal{R}_1 f_{E_0} - \frac{1}{3} (\alpha \varepsilon \mathcal{R}_1 \chi_D)^2 \right\}$.

2-1. Wave front set of distributions

Definition 3 (wave front set)

Let X be a manifold. For $u \in \mathcal{D}'(X)$ and $(x, \xi) \in T^*X \setminus \mathbf{0}$, we say that $(x, \xi) \notin \mathbf{WF}(u)$ if there exists $\phi \in \mathbf{C}_0^\infty(X)$ with $\phi(y) \neq \mathbf{0}$ near $y = x$ and a conic neighborhood of V at $\eta = \xi$ such that

$$\widehat{\phi u}(\eta) = \mathcal{O}(\langle \eta \rangle^{-M}) \quad \text{for } \eta \in V, \quad M = 1, 2, 3, \dots$$

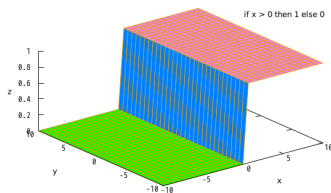
In other words,

$$\Sigma_x(u) := \bigcap_{\phi(x) \neq 0} \{\eta \in T_x^*X \setminus \mathbf{0} : \widehat{\phi u}(\eta) \neq \mathcal{O}(\langle \eta \rangle^{-\infty})\},$$

$$\mathbf{WF}(u) := \{(x, \xi) \in T^*X \setminus \mathbf{0} : \xi \in \Sigma_x(u)\}.$$

2-2. The characteristic function of a half-plane

Set $f(\mathbf{x}, \mathbf{y}) = 1$ if $\mathbf{x} > \mathbf{0}$, $\mathbf{0}$ otherwise in \mathbb{R}^2 :



Let $\phi(\mathbf{x}, \mathbf{y}) \in \mathcal{D}(\mathbb{R}^2)$.

- If $\phi(\mathbf{0}, \mathbf{y}) = \mathbf{0}$, then $\widehat{\phi f}(\zeta, \eta) = \hat{\phi}(\zeta, \eta)$ or $\widehat{\phi f}(\zeta, \eta) = \mathbf{0}$. We have $\widehat{\phi f}(\zeta, \eta) = \mathcal{O}(\langle \zeta; \eta \rangle^{-\infty})$, where $\langle \zeta; \eta \rangle = \sqrt{1 + |\zeta|^2 + |\eta|^2}$.
- If $\phi(\mathbf{0}, \mathbf{y}) \neq \mathbf{0}$, then

$$\widehat{\phi f}(\zeta, \eta) = \text{const} \cdot \text{pv} \int_{\mathbb{R}} \frac{\hat{\phi}(\zeta, \eta)}{\xi - \zeta} d\zeta.$$

- Then $\mathbf{WF}(f) = \{(\mathbf{0}, \mathbf{y}, \zeta, \mathbf{0}) \mid \mathbf{y}, \zeta \in \mathbb{R}, \zeta \neq \mathbf{0}\}$.

2-3. Conormal distributions

Definition 4 (Conormal distributions)

Let X be an N -dimensional manifold, and let Y be a closed submanifold of X . $u \in \mathcal{D}'(X)$ is said to be conormal with respect to Y of degree m if

$$L_1 \cdots L_M u \in {}^\infty H_{(-m-N/4)}^{\text{loc}}(X)$$

for all $M = 0, 1, 2, \dots$ and all vector fields L_1, \dots, L_M tangential to Y . Denote by $I^m(X, N^*Y)$ or $I^m(N^*Y)$, the set of all distributions on X conormal with respect to Y of degree m .

$$\|u\|_{\infty H_{(s)}(\mathbb{R}^N)} := \sup_{j=0,1,2,\dots} \left(\int_{X_j} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2},$$

$$X_0 := \{|\xi| < 1\}, \quad X_j := \{2^{j-1} \leq |\xi| < 2^j\}, \quad (j = 1, 2, 3, \dots).$$

2-4. Conormal distributions and oscillatory integrals

Proposition 5 (Characterization of conormal distributions)

Let X be an N -dimensional manifold, and let Y be a closed submanifold of X with $\text{codim } Y = k$. Fix arbitrary $y \in Y$ and choose local coordinates $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ such that $\mathbf{x}(y) = \mathbf{0}$ and Y is represented $\{\mathbf{x}' = \mathbf{0}\}$ locally. For $\mathbf{u} \in I^{m+k/2-N/4}(N^*Y)$ there exists an amplitude $\mathbf{a}(\mathbf{x}'', \zeta') \in \mathbf{S}^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ such that

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^k} \mathbf{e}^{i\mathbf{x}' \cdot \zeta'} \mathbf{a}(\mathbf{x}'', \zeta') d\zeta' \quad \text{near } \mathbf{x} = \mathbf{0}.$$

Here $\mathbf{S}^m(\mathbb{R}^{N-k} \times \mathbb{R}^k)$ is the set of all smooth functions on $\mathbb{R}^{N-k} \times \mathbb{R}^k$ such that for any compact set K in \mathbb{R}^{N-k} and multi-indices α, β

$$|\partial_{\mathbf{x}''}^{\beta} \partial_{\zeta'}^{\alpha} \mathbf{a}(\mathbf{x}'', \zeta')| \leq \mathbf{C}_{K\alpha\beta} \langle \zeta \rangle^{m-|\alpha|} \quad \text{for } (\mathbf{x}'', \zeta') \in K \times \mathbb{R}^k$$

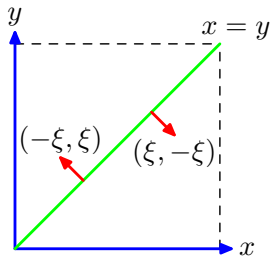
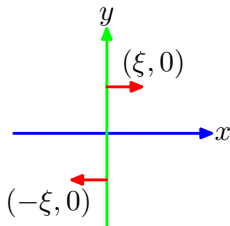
with some positive constant $\mathbf{C}_{K\alpha\beta}$.

2-5. Examples of conormal distributions

- $f(\mathbf{x}, \mathbf{y}) \in I^{-1}(\mathbf{N}^*(\{\mathbf{x} = \mathbf{0}\}))$.
- If $\mathbf{a}(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{S}^m(\mathbb{R}^N \times \mathbb{R}^N)$, then

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^N} \mathbf{e}^{i(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\zeta}} \mathbf{a}(\mathbf{x}, \boldsymbol{\zeta}) d\boldsymbol{\zeta} \in I^m(\mathbf{N}^* \Delta),$$

where $\Delta = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^N\}$.



2-6. non-degenerate phase functions

Definition 6 (non-degenerate phase functions)

Let \mathbf{X} be an N -dimensional manifold. We say that $\varphi(\mathbf{x}, \theta) \in \mathbf{C}^\infty(\mathbf{X} \times \mathbb{R}^k \setminus \{\mathbf{0}\}; \mathbb{R})$ is a non-degenerate phase function if

- $\varphi(\mathbf{x}, t\theta) = t\varphi(\mathbf{x}, \theta)$ for $(\mathbf{x}, \theta) \in \mathbf{X} \times \mathbb{R}^k \setminus \{\mathbf{0}\}$ and $t > 0$.
- $d\varphi(\mathbf{x}, \theta) \neq \mathbf{0}$.
- If $\varphi'_\theta(\mathbf{x}, \theta) = \mathbf{0}$, then $\text{rank}[d\varphi'_\theta(\mathbf{x}, \theta)] \equiv k$.

In this case $\mathbf{C}_\varphi = \{(\mathbf{x}, \theta) : \varphi'_\theta(\mathbf{x}, \theta) = \mathbf{0}\}$ is an N -dimensional submanifold of $\mathbf{X} \times \mathbb{R}^k \setminus \{\mathbf{0}\}$. Set

$\Lambda_\varphi := \{(\mathbf{x}, \varphi'_\mathbf{x}(\mathbf{x}, \theta)) : \varphi'_\theta(\mathbf{x}, \theta) = \mathbf{0}\}$. Then

$$\mathbf{C}_\varphi \ni (\mathbf{x}, \theta) \mapsto (\mathbf{x}, \varphi'_\mathbf{x}(\mathbf{x}, \theta)) \in \Lambda_\varphi$$

is a diffeomorphism and Λ_φ becomes a conic Lagrangian submanifold of $\mathbf{T}^*\mathbf{X} \setminus \mathbf{0}$, that is, a conic submanifold with $d\xi \wedge d\mathbf{x} \equiv \mathbf{0}$.

2-7. Lagrangian distributions

The notion of conormal distributions is generalized as follows.

Definition 7 (Lagrangian distribution)

Let Λ be a conic Lagrangian submanifold of $T^*\mathbf{X} \setminus \mathbf{0}$.

$I^m(\mathbf{X}, \Lambda) = I^m(\Lambda)$ is the set of all $\mathbf{u} \in \mathcal{D}'(\mathbf{X})$ satisfying

- $\mathbf{WF}(\mathbf{u}) \subset \Lambda$.
- For any $(\mathbf{x}_0, \zeta_0) \in \Lambda$ there exist a non-degenerate phase function $\varphi(\mathbf{x}, \theta)$ and an amplitude $\mathbf{a}(\mathbf{x}, \theta) \in \mathbf{S}^{m+N/4-k/2}(\mathbf{X} \times \mathbb{R}^k)$ such that $\Lambda = \Lambda_\varphi$ and

$$\mathbf{u}(\mathbf{x}) = \int \mathbf{e}^{i\varphi(\mathbf{x}, \theta)} \mathbf{a}(\mathbf{x}, \theta) d\theta \quad \text{near } (\mathbf{x}_0, \zeta_0).$$

2-8. Fourier integral operators

Let \mathbf{X} and \mathbf{Y} be manifolds, and let Λ be a conic Lagrangian submanifold of $\mathbf{T}^*(\mathbf{X} \times \mathbf{Y}) \setminus \mathbf{0}$. For $\mathbf{K}(\mathbf{x}, \mathbf{y}) \in I^m(\mathbf{X} \times \mathbf{Y}, \Lambda)$,

$$\mathbf{A}u(\mathbf{x}) := \int_{\mathbf{Y}} \mathbf{K}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y}, \quad u \in \mathcal{D}(\mathbf{Y})$$

defines a linear operator of $\mathcal{D}(\mathbf{Y})$ to $\mathcal{E}(\mathbf{X})$. \mathbf{A} is said to be a Fourier integral operator. Locally \mathbf{A} is given by

$$\mathbf{A}u(\mathbf{x}) := \iint e^{i\varphi(\mathbf{x}, \mathbf{y}, \theta)} \mathbf{a}(\mathbf{x}, \mathbf{y}, \theta) u(\mathbf{y}) d\mathbf{y} d\theta.$$

with some $\varphi(\mathbf{x}, \mathbf{y}, \theta)$ and $\mathbf{a}(\mathbf{x}, \mathbf{y}, \theta)$. If the principal part of \mathbf{a} does not vanish, then \mathbf{A} is called an elliptic FIO.

$\Lambda' := \{(\mathbf{x}, \mathbf{y}; \xi, \eta) : (\mathbf{x}, \mathbf{y}; \xi, -\eta) \in \Lambda\}$ is said to be the canonical relation of \mathbf{A} . We have $\mathbf{WF}(\mathbf{A}u) \subset \Lambda' \circ \mathbf{WF}(u)$ and if \mathbf{A} is elliptic then $\mathbf{WF}(\mathbf{A}u) = \Lambda' \circ \mathbf{WF}(u)$ holds.

3-1. $T^*G(d, n)$

Lemma 8

$T^*G(d, n)$ is expressed as

$$\{(\sigma, \mathbf{x}''; \eta_1'', \dots, \eta_d'', \zeta'') : \sigma \in G_{d,n}, \mathbf{x}'', \eta_1'', \dots, \eta_d'', \zeta'' \in \sigma^\perp\}.$$

Proof. Fix an arbitrary $(\sigma, \mathbf{x}'') \in G(d, n)$. The set of (co-)tangent vectors corresponding to σ is a $d(n-d)$ -dimensional vector space since $\sigma \in G_{d,n}$. There exists an orthonormal system $\{\omega_1, \dots, \omega_d\} \subset S^{n-1}$ such that $\sigma = \text{span}\langle \omega_1, \dots, \omega_d \rangle$. If we replace ω_j by $\mathbf{c}\omega_j$ with $\mathbf{c} \in \mathbb{R} \setminus \{0\}$, σ does not change. Hence (co-)tangent vector η_j corresponding to ω_j belongs to ω_j^\perp . Since σ is independent of the choice of $\{\omega_1, \dots, \omega_d\} \subset S^{n-1}$, (co-)tangent vectors η_1, \dots, η_d corresponding to σ belong $\omega_1^\perp \cap \dots \cap \omega_d^\perp = \sigma^\perp$. \square

Notation. The orthogonal projection of \mathbb{R}^n onto $\sigma \in G_{d,n}$ is denoted by π_σ .

3-2. The canonical relation of the d -plane transform

Theorem 9

\mathcal{R}_d is an elliptic Fourier integral operator whose distribution kernel belongs to

$$I^{0+(n-d)/2-(N(d,n)+n)/4}(\mathbf{G}(d, n) \times \mathbb{R}^n; \Lambda_\phi),$$

$$m + \frac{k}{2} - \frac{N}{4} = 0 + \frac{n-d}{2} - \frac{N(d, n) + n}{4} = -\frac{d(n-d+1)}{4}.$$

$$\Lambda'_\phi = \{(\sigma, \mathbf{y} - \pi_\sigma \mathbf{y}, \mathbf{y}; \pm |\pi_\sigma \mathbf{u}| \eta, \mathbf{0}, \dots, \mathbf{0}, \eta, \boldsymbol{\eta}) :$$

$$\sigma \in \mathbf{G}_{d,n}, \mathbf{y} \in \mathbb{R}^n, \eta \in \sigma^\perp\}$$

$$= \{(\sigma, \mathbf{x}'', \mathbf{x}'' + t\omega; t\zeta, \mathbf{0}, \dots, \mathbf{0}, \zeta, \boldsymbol{\xi}) :$$

$$(\sigma, \mathbf{x}'') \in \mathbf{G}(d, n), t \in \mathbb{R}, \omega \in \sigma \cap S^{-1}, \zeta \eta \in \sigma^\perp\}.$$

3-3. Fourier slice theorem

For any $\sigma \in \mathbf{G}_{d,n}$ and $\xi \in \sigma^\perp$,

$$\begin{aligned} \int_{\sigma^\perp} \mathbf{e}^{-i\mathbf{x}'' \cdot \xi} \mathcal{R}_d \mathbf{f}(\sigma, \mathbf{x}'') \, d\mathbf{x}'' &= \int_{\sigma^\perp} \int_{\sigma} \mathbf{e}^{-i\mathbf{x}'' \cdot \xi} \mathbf{f}(\mathbf{x}' + \mathbf{x}'') \, d\mathbf{x}' \, d\mathbf{x}'' \\ &= \int_{\sigma^\perp} \int_{\sigma} \mathbf{e}^{-i(\mathbf{x}' + \mathbf{x}'') \cdot \xi} \mathbf{f}(\mathbf{x}' + \mathbf{x}'') \, d\mathbf{x}' \, d\mathbf{x}'' = \hat{\mathbf{f}}(\xi''). \end{aligned}$$

The for any $(\sigma, \mathbf{x}'') \in \mathbf{G}(d, n)$,

$$\begin{aligned} \mathcal{R}_d \mathbf{f}(\sigma, \mathbf{x}'') &= \frac{1}{(2\pi)^{n-d}} \int_{\sigma^\perp} \mathbf{e}^{i\mathbf{x}'' \cdot \xi} \hat{\mathbf{f}}(\xi'') \, d\xi \\ &= \frac{1}{(2\pi)^{n-d}} \int_{\sigma^\perp} \int_{\mathbb{R}^n} \mathbf{e}^{i(\mathbf{x}'' - \mathbf{y}) \cdot \xi} \mathbf{f}(\mathbf{y}) \, d\mathbf{y} \, d\xi \\ &= \frac{1}{(2\pi)^{n-d}} \int_{\sigma^\perp} \int_{\mathbb{R}^n} \mathbf{e}^{i\phi(\sigma, \mathbf{x}'', \mathbf{y}, \xi)} \mathbf{f}(\mathbf{y}) \, d\mathbf{y} \, d\xi, \end{aligned}$$

$$\phi(\sigma, \mathbf{x}'', \mathbf{y}, \xi) = (\mathbf{x}'' - \mathbf{y}) \cdot \xi, \quad (\sigma, \mathbf{x}'') \in \mathbf{G}(d, n), \mathbf{y} \in \mathbb{R}^n, \xi \in \sigma^\perp.$$

It suffices to show that ϕ is a non-degenerate phase function.

3-4. $\phi(\sigma, \mathbf{x}'', \mathbf{y}, \eta)$ is a phase function.

- Critical points $\phi'_{\zeta} = \mathbf{0}$:

For $((\sigma, \mathbf{x}''), \mathbf{y}, \Xi) \in \mathbf{G}(\mathbf{d}, \mathbf{n}) \times \mathbb{R}^n \times \mathbb{R}^n$, set $\Psi_1(\sigma, \mathbf{x}'', \mathbf{y}, \Xi) := \phi(\sigma, \mathbf{x}'', \mathbf{y}, \Xi - \pi_{\sigma}\Xi)$. Then

$$\Psi_1 = (\mathbf{x}'' - \mathbf{y}) \cdot (\Xi - \pi_{\sigma}\Xi) = (\mathbf{x}'' - \mathbf{y} + \pi_{\sigma}\mathbf{y}) \cdot \Xi,$$

$$\phi'_{\zeta} = \nabla_{\Xi}\Psi_1|_{\Xi=\zeta} = \mathbf{x}'' - (\mathbf{y} - \pi_{\sigma}\mathbf{y}).$$

- Non-degeneracy $\text{rank } \mathbf{d}\phi'_{\zeta} \equiv n - \mathbf{d}$:

$\phi''_{\zeta\mathbf{y}} = -I_n + \pi_{\sigma}$ implies $\text{rank } \phi''_{\zeta\mathbf{y}} \equiv n - \mathbf{d}$ and $\text{rank } \mathbf{d}\phi'_{\zeta} \equiv n - \mathbf{d}$.

- non-vanishing $\mathbf{d}\phi \neq \mathbf{0}$ for $\zeta \neq \mathbf{0}$:

Since $\phi'_{\mathbf{y}} = \zeta \neq \mathbf{0}$ for $\zeta \neq \mathbf{0}$, $\mathbf{d}\phi \neq \mathbf{0}$ for $\zeta \neq \mathbf{0}$.

3-5. Λ'_ϕ

Let us express $\sigma \in \mathbf{G}_{d,n}$ by $\sigma = \mathbf{span}\langle \omega_1, \dots, \omega_d \rangle$. By using the similar arguments in the previous page, we have at the critical points

$$\mathbf{x}'' = \mathbf{y} - \pi_\sigma \mathbf{y}$$

$$(\sigma, \mathbf{x}'', \mathbf{y}; \phi'_{\sigma, \mathbf{x}''}, -\phi'_\mathbf{y}) = (\sigma, \mathbf{x}'', \mathbf{y}; (\mathbf{y} \cdot \omega_1, \dots, \mathbf{y} \cdot \omega_d, \mathbf{1}) \xi, \xi).$$

By using a rotation on σ , we can choose $\{\omega_1, \dots, \omega_d\}$ such that

$$\mathbf{y} \cdot \omega_1 = \pm |\pi_\sigma \mathbf{y}|, \quad \mathbf{y} \cdot \omega_2 = \dots = \mathbf{y} \cdot \omega_d = \mathbf{0}.$$

We have at the critical points $\mathbf{x}'' = \mathbf{y} - \pi_\sigma \mathbf{y}$

$$(\sigma, \mathbf{x}'', \mathbf{y}; \phi'_{\sigma, \mathbf{x}''}, -\phi'_\mathbf{y}) = (\sigma, \mathbf{x}'', \mathbf{y}; (\pm |\pi_\sigma \mathbf{y}|, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}) \xi, \xi).$$

This completes the proof of Theorem 9. □

4-1. Main Theorem

- Suppose that the metal region $\mathbf{D} \subset \mathbb{R}^n$ is a disjoint union of \mathbf{D}_j ($j = 1 \dots, \mathbf{J}$) which are simply connected, strictly convex and bounded with smooth boundaries $\Sigma_j := \partial \mathbf{D}_j$. Set $\Sigma := \partial \mathbf{D}$.
- For $j \neq k$ \mathcal{L}_{jk} denotes the set of hyperplanes tangential to Σ_j and Σ_k . Set $\mathcal{L} := \cup \mathcal{L}_{jk}$.
- $\chi_{\mathbf{D}_j} \in I^{-1+1/2-n/4}(\mathbf{N}^* \Sigma_j)$, $\chi_{\mathbf{D}} \in I^{-1+1/2-n/4}(\mathbf{N}^* \Sigma)$ (c.f. $\mathbf{f}(\mathbf{x}, \mathbf{y})$).

Theorem 10

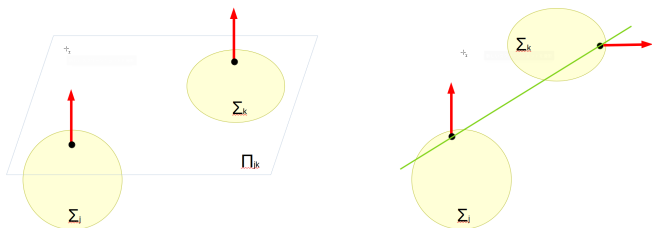
Away from Σ , the nonlinear part of the CT image

$$\mathbf{f}_{MA} := \mathbf{f}_{CT} - \mathbf{f}_{E_0} = \sum_{k=1}^{\infty} \mathbf{A}_k(\mathbf{a}\varepsilon)^{2k} \mathcal{R}_d^*(-\Delta_{\mathbf{x}''})^{d/2} [(\mathcal{R}_d \chi_{\mathbf{D}})^{2k}]$$

belongs to $I^{-(d+3)/2+1/2-n/4}(\mathbf{N}^ \mathcal{L})$, and its principal symbol does not vanish. This means that \mathbf{f}_{MA} causes the metal streaking artifacts.*

4-2. Illustration of Theorem 10

- Park-Choi-Seo (CPAM, 2017) proved that $\mathbf{WF}(\mathbf{F}_{MA}) = \mathbf{N}^* \mathcal{L}$ for $(\mathbf{n}, \mathbf{d}) = (\mathbf{2}, \mathbf{1})$.
- Theorem 10 is the generalization of Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) for $(\mathbf{n}, \mathbf{d}) = (\mathbf{2}, \mathbf{1})$.
- If Σ_j and Σ_k have a common tangent hyperplane Π_{jk} , then their microlocal singularities, which are in the same conormal direction, spread all over Π_{jk} .
- If Σ_j and Σ_k have a common tangent \mathbf{d} -plane with $\mathbf{d} \leq \mathbf{n} - \mathbf{2}$ and the directions of their microlocal singularities are different, then no nonlinear effect occurs.



4-3. The canonical transform of D

We need to consider

$$(\mathcal{R}_d \chi_D)^2 = \sum_{j=1}^J (\mathcal{R}_d \chi_{D_j})^2 + 2 \sum_{1 \leq j < k \leq J} \mathcal{R}_d \chi_{D_j} \cdot \mathcal{R}_d \chi_{D_k},$$

$$\Lambda'_\phi \circ N^* \Sigma_j = \{ (\sigma, \mathbf{y} - \pi_\sigma \mathbf{y}, \mathbf{y}; (\pm |\pi_\sigma \mathbf{y}|, \mathbf{0}, \dots, \mathbf{0}, 1) \eta) : (\mathbf{y}, \eta) \in N^* \Sigma_j \setminus \mathbf{0}, \sigma \in \mathbf{G}_{d,n}, \sigma \subset \eta^\perp \}.$$

Set

$$\begin{aligned} \mathbf{S}_j &:= \pi_{\mathbf{G}(d,n)} (\Lambda'_\phi \circ N^* \Sigma_j) \\ &= \{ (\sigma, \mathbf{y} - \pi_\sigma \mathbf{y}) : \mathbf{y} \in \Sigma_j \setminus \mathbf{0}, \sigma \in \mathbf{G}_{d,n}, \sigma \subset T_{\mathbf{y}} \Sigma_j \}. \end{aligned}$$

Lemma 11

- $\text{codim } \mathbf{S}_j = 1$, and $N^* \mathbf{S}_j = \Lambda'_\phi \circ N^* \Sigma_j$.
- If $j \neq k$ and $\mathbf{S}_j \cap \mathbf{S}_k \neq \emptyset$, then \mathbf{S}_j intersects \mathbf{S}_k transversally.

4-4. $\mathbf{S}_j \cap \mathbf{S}_k$ and its conormal bundle

If $(\sigma, \mathbf{x}'') \in \mathbf{S}_j \cap \mathbf{S}_k$, then there exist $\mathbf{y}_j \in \Sigma_j$ and $\mathbf{y}_k \in \Sigma_k$ such that $\sigma \subset T_{\mathbf{y}_j} \Sigma_j \cap T_{\mathbf{y}_k} \Sigma_k$ and $\mathbf{x}'' = \mathbf{y}_j - \pi_\sigma \mathbf{y}_j = \mathbf{y}_k - \pi_\sigma \mathbf{y}_k$. Using the same notation we set for $l = d, \dots, n-1$

$$(\mathbf{S}_j \cap \mathbf{S}_k)_l := \{(\sigma, \mathbf{x}'') \in \mathbf{S}_j \cap \mathbf{S}_k : \dim(T_{\mathbf{y}_j} \Sigma_j \cap T_{\mathbf{y}_k} \Sigma_k) = l\}.$$

Lemma 12

- If $N_{\mathbf{y}_j}^* \Sigma_j = N_{\mathbf{y}_k}^* \Sigma_k$, then $T_{\mathbf{y}_j} \Sigma_j = T_{\mathbf{y}_k} \Sigma_k$ and $l = n-1$.
- $\text{codim}(\mathbf{S}_j \cap \mathbf{S}_k)_l = d(n-l-1) + 2$.

Set $\mathbf{S}_{jk} := (\mathbf{S}_j \cap \mathbf{S}_k)_{n-1}$. Then $\text{codim } \mathbf{S}_{jk} = 2$, and

$$N^* \mathbf{S}_{jk} = \left\{ (\sigma, \mathbf{x}''; (t, \mathbf{0}, \dots, \mathbf{0}, 1)\eta) : \right. \\ \left. (\sigma, \mathbf{x}'') \in \mathbf{S}_{jk}, t \in \mathbb{R}, \eta \in N_{\mathbf{y}_j}^* \Sigma_j \setminus \mathbf{0} \right\},$$

$$(t, \mathbf{0}, \dots, \mathbf{0}, 1)\eta = \frac{t_k - t}{t_k - t_j} (t_j, \mathbf{0}, \dots, \mathbf{0}, 1)\eta + \frac{t - t_j}{t_k - t_j} (t_k, \mathbf{0}, \dots, \mathbf{0}, 1)\eta.$$

4-5. Interaction near $(\mathbf{S}_j \cap \mathbf{S}_k)_I$ ($l < n - 1$)

Theorem 9 implies that $\mathcal{R}_{d\chi_{D_j}} \in I^{-(d+2)/2+1/2-N(d,n)/4}(\mathbf{N}^* \mathbf{S}_j)$. We show that **we can neglect the interaction near $(\mathbf{S}_j \cap \mathbf{S}_k)_I$ ($l < n - 1$)**. Fix an arbitrary $\Xi \in (\mathbf{S}_j \cap \mathbf{S}_k)_I$. We can choose local coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N(d,n)-2}$ such that $\Xi = \mathbf{0}$. $\mathbf{u} \in I^{-\mu}(\mathbf{N}^* \mathbf{S}_j)$ and $\mathbf{v} \in I^{-\mu}(\mathbf{N}^* \mathbf{S}_k)$ are given by

$$\mathbf{u} = \int_{\mathbb{R}} \mathbf{e}^{ix\xi} \mathbf{a}(\mathbf{y}, \mathbf{z}, \xi) d\xi, \quad \mathbf{v} = \int_{\mathbb{R}} \mathbf{e}^{iy\eta} \mathbf{b}(\mathbf{x}, \mathbf{z}, \eta) d\eta,$$

and since $\langle \xi \rangle^{-N} \langle \eta \rangle^{-N} \langle D_x \rangle^N \langle D_y \rangle^N \mathbf{e}^{i(x\xi+y\eta)} = \mathbf{e}^{i(x\xi+y\eta)}$

$$\begin{aligned} \mathbf{u}\mathbf{v} &= \iint_{\mathbb{R}^2} \mathbf{e}^{i(x\xi+y\eta)} \mathbf{a}(\mathbf{y}, \mathbf{z}, \xi) \mathbf{b}(\mathbf{x}, \mathbf{z}, \eta) d\xi d\eta \\ &= \iint_{\mathbb{R}^2} \mathbf{e}^{i(x\xi+y\eta)} \frac{\langle D_y \rangle^N \mathbf{a}(\mathbf{y}, \mathbf{z}, \xi)}{\langle \xi \rangle^N} \frac{\langle D_x \rangle^N \mathbf{b}(\mathbf{x}, \mathbf{z}, \eta)}{\langle \eta \rangle^N} d\xi d\eta \end{aligned}$$

for any $N > 0$ near 0 .

4-6. Paired Lagrangian distributions

Definition 13 (Paired Lagrangian distributions)

Let $\mu, \nu \in \mathbb{R}$, and let Λ_0, Λ_1 be conic Lagrangian submanifold of $T^*X \setminus \mathbf{0}$. We say that $u \in \mathcal{D}'(X)$ belongs to $I^{\mu, \nu}(\Lambda_0, \Lambda_1)$ if $\text{WF}(u) \subset \Lambda_0 \cup \Lambda_1$ and **away from** $\Lambda_0 \cap \Lambda_1$

$$u \in I^{\mu+\nu}(\Lambda_0 \setminus \Lambda_1), \quad u \in I^{\mu}(\Lambda_1).$$

Lemma 14 (Greenleaf-Uhlmann, 1993)

Let X be an N -dimensional manifold, and let Y, Z be submanifolds of X with $\text{codim } Y = k_1$ and $\text{codim } Z = l_1$ respectively. Suppose $Y \pitchfork Z$ and set $\text{codim } Y \cap Z = k_1 + k_2 = l_1 + l_2$. Then

$$\begin{aligned} & I^{\mu+k_1/2-N/4}(N^*Y) \cdot I^{\nu+k_1/2-N/4}(N^*Z) \\ & \subset I^{\mu+k_1/2-N/4, \nu+k_2/2}(N^*(Y \cap Z), N^*Y) \\ & + I^{\nu+l_1/2-N/4, \mu+l_2/2}(N^*(Y \cap Z), N^*Z). \end{aligned}$$

4-7. Outline of Proof of Theorem 10

- Set $\mathcal{A} := \sum_{j \neq k} I^{-(d+2)/2+1/2-N(d,b)/4, -(d+2)/2+1/2} (N^* S_{jk}, N^* S_j)$.
- Note that $I^{-(d+2)/2+1/2-N(d,b)/4} (N^* S_j) \subset \mathcal{A}$.
- Lemma 14 proves that $(\mathcal{R}_d \chi_D)^2 \in \mathcal{A}$.
- It follows that \mathcal{A} is an algebra. In particular

$$P_{MA} := \sum_{k=1}^{\infty} A_k (\alpha \varepsilon)^{2k} (\mathcal{R}_d \chi_D)^{2k} \in \mathcal{A}.$$
- Applying Lemma 15 to P_{MA} , we prove Theorem 10.

Lemma 15

$\mathcal{R}_d^*(-\Delta_{x''})^{d/2}$ is a FIO of order $\frac{n + N(d, n)}{4} - \frac{n - d}{2}$ with a canonical relation $(\Lambda'_\phi)^* := \{(\mathbf{x}, \mathbf{y}, \xi, \eta) : (\mathbf{y}, \mathbf{x}, \eta, \xi) \in \Lambda'_\phi\}$, and $(\Lambda'_\phi)^* \circ N^* S_j = N^* \Sigma_j$, $(\Lambda'_\phi)^* \circ N^* S_{jk} = N^* \mathcal{L}_{jk}$.

4-8. $I^{\mu, \nu} (N^* \mathbf{S}_{jk}, N^* \mathbf{S}_j)$ is given by oscillatory integrals.

If $\mathbf{u} \in I^{-(d+2)/2+1/2-N(d,b)/4, -(d+2)/2+1/2} (N^* \mathbf{S}_{jk}, N^* \mathbf{S}_j)$, we can choose local coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N(d,n)-2}$ and find $\mathbf{a}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta)$ such that $\mathbf{S}_j = \{\mathbf{x} = \mathbf{0}\}$, $\mathbf{S}_{jk} = \{\mathbf{x} = \mathbf{y} = \mathbf{0}\}$,

$$\partial_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^\gamma \partial_\zeta^\alpha \partial_\eta^\beta \mathbf{a}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta) = \mathcal{O}(\langle \zeta; \eta \rangle^{-(d+2)/2-\alpha} \langle \eta \rangle^{-(d+2)/2-\beta}),$$

$$\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \iint_{\mathbb{R}^2} \mathbf{e}^{i(x\zeta+y\eta)} \mathbf{a}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta) d\zeta d\eta$$

near $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$. Such a class of symbols is denoted by \mathcal{B}_d . We shall evaluate the products of such paired Lagrangian distributions. To show that \mathcal{A} is an algebra, we shall check that \mathbf{U}_d is closed under convolutions in ζ and η .

Pick up $\psi(t) \in \mathbf{C}^\infty(\mathbb{R})$ such that $\mathbf{0} \leq \psi(t) \leq \mathbf{1}$,
 $\psi(t) = \mathbf{1}$ for $|t| \leq \mathbf{1}/\mathbf{2}$, $\mathbf{0} < \psi(t) < \mathbf{1}$ for $\mathbf{1}/\mathbf{2} < |t| < \mathbf{3}/\mathbf{4}$,
 $\psi(t) = \mathbf{0}$ for $|t| \geq \mathbf{3}/\mathbf{4}$. Then $\mathbf{1} - \psi(t) = \mathbf{1}$ for $|t| \geq \mathbf{3}/\mathbf{4}$,
 $\mathbf{0} < \mathbf{1} - \psi(t) < \mathbf{1}$ for $\mathbf{1}/\mathbf{2} < |t| < \mathbf{3}/\mathbf{4}$, $\mathbf{1} - \psi(t) = \mathbf{0}$ for $|t| \leq \mathbf{1}/\mathbf{2}$.

4-9. $(I^{\mu, \nu} (N^* S_{jk}, N^* S_j))^2 \subset \mathcal{A}$

Suppose that there exists $\mathbf{b}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta) \in \mathbf{U}_d$ such that

$$\mathbf{v}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \iint_{\mathbb{R}^2} \mathbf{e}^{i(\mathbf{x}\zeta + \mathbf{y}\eta)} \mathbf{b}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta) d\zeta d\eta.$$

Then

$$\mathbf{u}\mathbf{v} = \iint_{\mathbb{R}^2} \mathbf{e}^{i(\mathbf{x}\zeta + \mathbf{y}\eta)} \mathbf{c}_1(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta, \eta) d\zeta d\eta,$$

$$\mathbf{c} = \iint_{\mathbb{R}^2} \mathbf{a}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \zeta - \zeta', \eta - \lambda) \mathbf{b}(\mathbf{x}, \mathbf{y}, \zeta, \lambda) d\zeta' d\lambda.$$

It suffices to show that $\mathbf{c}_1 \in \mathbf{U}_d$.

$$\text{Set } \Phi(\zeta, \eta, \zeta', \lambda) := \psi\left(\frac{\langle \zeta - \zeta', \eta - \lambda \rangle}{\langle \zeta; \eta \rangle}\right), \Psi(\eta, \lambda) := \psi\left(\frac{\langle \eta - \lambda \rangle}{\langle \eta \rangle}\right).$$

- $|(\zeta, \eta)| \sim |(\zeta', \lambda)|$ for $\Phi > 0$.
- $\langle \zeta - \zeta', \eta - \lambda \rangle \geq \langle \zeta; \eta \rangle / 2$ for $1 - \Phi > 0$.
- $\partial_{\zeta'}^{\alpha} \partial_{\lambda}^{\beta} \Phi = \mathcal{O}(\langle \zeta - \zeta', \eta - \lambda \rangle^{-\alpha - \beta})$.

4-10. $\mathbf{c}_1 \in \mathbf{U}_d$

Note that $\partial_{\xi}(\mathbf{g}(\xi - \zeta)) = -\partial_{\zeta}(\mathbf{g}(\xi - \zeta))$. Using

$$\mathbf{1} = \Phi\Psi + (\mathbf{1} - \Phi)\Psi + \Phi(\mathbf{1} - \Psi) + (\mathbf{1} - \Phi)(\mathbf{1} - \Psi),$$

we deduce that

$$\begin{aligned} \partial_{\mathbf{x}, \mathbf{y}, \mathbf{z}}^{\gamma} \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \mathbf{c} &= \sum_{\gamma' \leq \gamma} \mathbf{C}(\gamma') \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{\alpha, \beta} \mathbf{b}_{\gamma - \gamma'} d\zeta d\lambda \\ \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{\alpha, \beta} \mathbf{b}_{\gamma - \gamma'} d\zeta d\lambda &= \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{\alpha, \beta} \partial_{\xi}^{\alpha} \partial_{\lambda}^{\beta} (\Phi\Psi \mathbf{b}_{\gamma - \gamma'}) d\zeta d\lambda \\ &\quad + \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{\alpha, 0} \partial_{\lambda}^{\beta} ((\mathbf{1} - \Phi)\Psi \mathbf{b}_{\gamma - \gamma'}) d\zeta d\lambda \\ &\quad + \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{0, \beta} \partial_{\xi}^{\alpha} (\Phi(\mathbf{1} - \Psi) \mathbf{b}_{\gamma - \gamma'}) d\zeta d\lambda \\ &\quad + \iint_{\mathbb{R}^2} \mathbf{a}_{\gamma'}^{\alpha, \beta} (\mathbf{1} - \Phi)(\mathbf{1} - \Psi) \mathbf{b}_{\gamma - \gamma'} d\zeta d\lambda \\ &= \mathcal{O}(\langle \xi; \eta \rangle^{-(d+2)/2 - \alpha} \langle \eta \rangle^{-(d+2)/2 - \beta}). \end{aligned}$$

$$4-11. I^{\mu, \nu} (N^* S_{jk}, N^* S_j) \cdot I^{\mu, \nu} (N^* S_{jk}, N^* S_k) \subset \mathcal{A}$$

Suppose that there exists $\mathbf{b}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in \mathbf{U}_d$ such that

$$\mathbf{v}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \iint_{\mathbb{R}^2} \mathbf{e}^{i(\mathbf{x}\boldsymbol{\zeta} + \mathbf{y}\boldsymbol{\eta})} \mathbf{b}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\zeta}) d\boldsymbol{\zeta} d\boldsymbol{\eta}.$$

Then

$$\mathbf{uv} = \iint_{\mathbb{R}^2} \mathbf{e}^{i(\mathbf{x}\boldsymbol{\zeta} + \mathbf{y}\boldsymbol{\eta})} \mathbf{p}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\zeta}, \boldsymbol{\eta}) d\boldsymbol{\zeta} d\boldsymbol{\eta},$$

$$\mathbf{p} = \iint_{\mathbb{R}^2} \mathbf{a}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\zeta} - \boldsymbol{\zeta}, \boldsymbol{\eta} - \boldsymbol{\lambda}) \mathbf{b}(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\zeta}) d\boldsymbol{\zeta} d\boldsymbol{\lambda}.$$

In the same way as in the previous page we have

$$\partial_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \partial_{\boldsymbol{\zeta}}^\alpha \partial_{\boldsymbol{\eta}}^\beta \mathbf{p} = \mathcal{O}(\langle \boldsymbol{\zeta} \rangle^{-(d+2)/2 - \alpha} \langle \boldsymbol{\eta} \rangle^{-(d+2)/2 - \beta}).$$

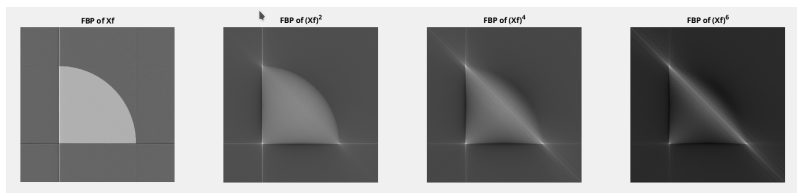
If we set

$$\mathbf{q}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\zeta}, \boldsymbol{\eta}) := \psi \left(\frac{\langle \boldsymbol{\zeta} \rangle}{\langle \boldsymbol{\eta} \rangle} \right) \mathbf{p}, \quad \mathbf{r}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\zeta}) := \mathbf{p} - \mathbf{q},$$

then $\mathbf{q}, \mathbf{r} \in \mathbf{U}_d$, which implies $I \cdot I' \subset I + I'$. □

5. Discussions

- $P_{MA} \notin \mathcal{R}_d(\mathcal{E}'(\mathbb{R}^n))$.
- Palacios-Uhlmann-Wang (SIAM J. Math. Anal., 2018) studied the case that D_1, \dots, D_J are convex polygons in \mathbb{R}^2 .



When D_1, \dots, D_J are convex polyhedra in \mathbb{R}^3 , the situation is more complicated.

- Theorem 9 might be useful to studying limited data problem.
- It might be worth studying these topics in the setting of some kind of Riemannian manifolds.

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