A note on the Bargmann transform on the Euclidean space

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Abstract

This note summarizes the basic facts on the Bargmann transform on the Euclidean space and some related topics. Recently these materials are used not only for pure mathematics like microlocal analysis, semiclassical analysis and functional analysis, but also for applied mathematics including image and signal processing.

1 Bargmann transform on the Euclidean space

In 1961 Bargmann introduced the global Bargmann transform on $\mathbb{R}^n$ and the Segal-Bargmann space of entire functions on $\mathbb{C}^n$ in his historical paper [1] to study the theory of quantization. Following Folland's textbook [3] mainly, we summarize the basic facts and related topics for this. The global Bargmann transform of a function $u(x)$ of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is defined by

$$\mathcal{B}u(z) := C_n \int_{\mathbb{R}^n} e^{-(z^2/4 - zx + x^2/2)} u(x) dx, \quad z \in \mathbb{C}^n,$$

where $C_n = 2^{-n/2}\pi^{-3n/4}$,

$$z\zeta := z_1\zeta_1 + \cdots + z_n\zeta_n \quad \text{for} \quad z = (z_1, \dots, z_n), \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n,$$

$z^2 := zz$, $|z| := \sqrt{z\bar{z}}$. The estimate

$$\left| e^{-(z^2/4 - zx + x^2/2)} \right| = e^{|z|^2/4 - (x - \mathrm{Re}(z))^2/2}$$

shows that $e^{-(z^2/4 - zx + x^2/2)}$ is a rapidly decreasing funtion of $x \in \mathbb{R}^n$ for any fixed $z \in \mathbb{C}^n$. So the glocal Bargmann transform can be defined for tempered distributions. To state this precisely we introduce some function spaces and notation. We denote by $L^2(\mathbb{R}^n)$ the set of all square intergrable functions on $\mathbb{R}^n$ with respect to the standard Lebesgue measure dx. Let $\mathrm{Hol}(\mathbb{C}^n)$ be the set of all holomorphic functions on $\mathbb{C}^n$. We denote by $L^2\big(\mathbb{C}^n, e^{-|z|^2/2} L(dz)\big)$ the set of all square intergrable functions on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ with respect to a weighted measure $e^{-|z|^2/2} L(dz)$, where $L(dz)$ is the standard Lebesgue measure on $\mathbb{C}^n$. The Schwartz class (the set of all rapidly decreasing functions) and its topological dual (the set of all tempered distributions) on $\mathbb{R}^n$ are denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ respectively. We set

$$\mathcal{F}_n := L^2\big(\mathbb{C}^n, e^{-|z|^2/2} L(dz)\big) \cap \mathrm{Hol}(\mathbb{C}^n)$$

for short. Then $\mathcal{F}_n$ is a closed subspace of $L^2\big(\mathbb{C}^n, e^{-|z|^2/2} L(dz)\big)$, and becomes a Hilbert space called Segal-Bargmann space, Bargmann-Fock space or Fock space. We now state the mapping properties of the global Bargmann transform.

Theorem 1. *The global Bargmann transform $\mathcal{B}$ is a linear continuous bijection in the following frameworks respectively*:

$$\begin{aligned}
&\mathcal{B}: \mathscr{S}(\mathbb{R}^n) \to \left(e^{|z|^2/4}\mathscr{S}(\mathbb{C}^n)\right) \cap \mathrm{Hol}(\mathbb{C}^n),\\
&\mathcal{B}: L^2(\mathbb{R}^n) \to \mathcal{F}_n,\\
&\mathcal{B}: \mathscr{S}'(\mathbb{R}^n) \to \left(e^{|z|^2/4}\mathscr{S}'(\mathbb{C}^n)\right) \cap \mathrm{Hol}(\mathbb{C}^n).
\end{aligned}$$

In particular $\mathcal{B}: \mathcal{B}: L^2(\mathbb{R}^n) \to \mathcal{F}_n$ is a Hilbert space isomorphism:

$$\langle \mathcal{B}u, \mathcal{B}v\rangle_{L^2\left(\mathbb{C}^n, e^{-|z|^2/2}L(dz)\right)} = \langle u, v\rangle_{L^2(\mathbb{R}^n)}, \quad u, v \in L^2(\mathbb{R}^n).$$

The adjoint $\mathcal{B}^*: \mathcal{F}_n \to L^2(\mathbb{R}^2)$ of $\mathcal{B}$ is given by

$$\mathcal{B}^*U(x) = C_n \int_{\mathbb{C}^n} e^{-(\bar{z}^2/4 - x\bar{z} + x^2/2)} U(z) e^{-|z|^2/2} L(dz). \tag{1}$$

The explicit computation in the framework of Schwartz classes gives $\mathcal{B}^* \circ \mathcal{B} = I_d$ and

$$\mathcal{B} \circ \mathcal{B}^* U(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} e^{z\bar{\zeta}} U(\zeta) e^{-|\zeta|^2/2} L(d\zeta). \tag{2}$$

We remark that for $U, V \in L^2\big(\mathbb{C}^n, e^{-|z|^2/2}L(dz)\big)$

$$\begin{aligned}
\left|e^{z\bar{\zeta}} U(\zeta) e^{-|\zeta|^2/2} \overline{V(z)} e^{-|z|^2/2}\right| &= e^{-|z-\zeta|^2/4} \cdot |U(\zeta)| e^{-|\zeta|^2/4} \cdot |V(z)| e^{-|z|^2/4}\\
&\in L^2\big(\mathbb{C}^n \times \mathbb{C}^n, L(d\zeta)L(dz)\big).
\end{aligned}$$

We have the following.

Theorem 2.

- *The adjoint $\mathcal{B}^*$ can be extended for $\mathcal{B}\big(\mathscr{S}'(\mathbb{R}^n)\big)$ and*

$$\mathcal{B}^* \circ \mathcal{B}u = u, \quad u \in \mathscr{S}(\mathbb{R}^n), \qquad \mathcal{B} \circ \mathcal{B}^* U = U, \quad U \in \mathcal{B}\big(\mathscr{S}'(\mathbb{R}^n)\big).$$

- *The equation* (2) *can be extended for $L^2\big(\mathbb{C}^n, e^{-|z|^2/2}L(dz)\big)$. In particular*

$$U(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{C}^n} e^{z\bar{\zeta}} U(\zeta) e^{-|\zeta|^2/2} L(d\zeta) \tag{3}$$

  holds for any $U \in \mathcal{F}_n$. So $\mathcal{F}_n$ is a reproducing kernel Hilbert space with a reproducing kernel $e^{z\bar{\zeta}}/(2\pi)^n$.

We apply the Taylor series of $e^{z\bar{\zeta}}$ to (3) to have

$$U(z) = \sum_{\alpha \in \mathbb{N}_0^n} \langle U, \varphi_\alpha\rangle_{\mathcal{F}_n} \varphi_\alpha(z), \quad \varphi_\alpha(z) = \frac{z^\alpha}{\sqrt{(2\pi)^n 2^{|\alpha|}\alpha!}},$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$,

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n, \quad z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Theorem 3. $\{\varphi_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ *is a complete orthonormal system of* $\mathcal{F}_n$.

$\{\mathcal{B}^* \varphi_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is a complete orthonormal system of $L^2(\mathbb{R}^n)$ since $\mathcal{B}^* : \mathcal{F}_n \to L^2(\mathbb{R}^n)$ is a Hilbert s@ace isomorphism. We say that $h_\alpha := \mathcal{B}^* \varphi_\alpha$ is the (normalized) α-th Hermite function.

2 Hermite functions

In this section we obtain the α-th Hermite functions $h_\alpha := \mathcal{B}^* \varphi_\alpha$, $\alpha \in \mathbb{N}_0^n$ concretely, and summarize their basic properties. To obtain $h_\alpha := \mathcal{B}^* \varphi_\alpha$, we make use of the following relations obtained by elementary computation.

Theorem 4. *Define unbounded linear operators* $A_j : \mathcal{F}_n \to \mathcal{F}_n$, $P_j : L^2(\mathbb{R}^n) rightarrow L^2(\mathbb{R}^2)$, $j = 1, \ldots, n$ *by*

$$A_j U(z) := 2 \frac{\partial U}{\partial z_j}(z), \quad U \in \mathcal{B}\big(\mathscr{S}(\mathbb{R}^n)\big), \quad P_j u(x) := \frac{\partial u}{\partial x_j}(x) + x_j u(x), \quad u \in \mathscr{S}(\mathbb{R}^n).$$

Then we have $A_j \circ \mathcal{B} = \mathcal{B} \circ P_j$ *and* $A_j^* \circ \mathcal{B} = \mathcal{B} \circ P_j^*$, *where*

$$A_j^* U(z) := z_j U(z), \quad U \in \mathcal{B}\big(\mathscr{S}(\mathbb{R}^n)\big), \quad P_j^* u(x) := -\frac{\partial u}{\partial x_j}(x) + x_j u(x), \quad u \in \mathscr{S}(\mathbb{R}^n).$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we set $A^\alpha := A_1^{\alpha_1} \ldots A_n^{\alpha_n}$. Similarly we define $(A^*)^\alpha$, P^α and $(P^*)^\alpha$. We remark that A^α, $(A^*)^\alpha$, P^α and $(P^*)^\alpha$ are independent of the order of the product and that $(A^*)\alpha = (A^\alpha)^*$ and $(P^*)^\alpha = (P^\alpha)^*$ hold since

$$[A_j, A_k] = 0, \quad [A_j^*, A_k^*] = 0, \quad [P_j, P_k] = 0, \quad [P_j^*, P_k^*] = 0,$$

for any $j, k = 1, \ldots, n$. It is easy to see that

$$h_0(x) = \mathcal{B}^* \varphi_0(x) = \pi^{-n/4} e^{-x^2/2}, \quad \varphi_\alpha(z) = \frac{(A^*)^\alpha \varphi_0(z)}{\sqrt{2^{|\alpha|} \alpha!}} = \frac{(A^*)^\alpha \circ \mathcal{B}^* h_0(z)}{\sqrt{2^{|\alpha|} \alpha!}}.$$

Then we deduce that

$$h_\alpha(x) = \frac{\mathcal{B}^* \circ (A^*)^\alpha \circ \mathcal{B} h_0(x)}{\sqrt{2^{|\alpha|} \alpha!}} = \frac{(P^*)^\alpha h_0(x)}{\sqrt{2^{|\alpha|} \alpha!}} = \frac{(-1)^{|\alpha|}}{\pi^{n/4} \sqrt{2^{|\alpha|} \alpha!}} \left(\frac{\partial}{\partial x} - x \right)^\alpha e^{-x^2/2}.$$

Applying the identity

$$\frac{df}{dt}(t) - t f(t) = e^{t^2/2} \frac{d}{dt}\big(e^{-t^2/2} f(t)\big)$$

to the above, we obtain the concrete formula of $h_\alpha(x)$.

Theorem 5. *$\{h_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is an orthonormal system of $L^2(\mathbb{R}^n)$ and given by the Rodrigues formula*

$$h_\alpha(x) = \frac{(-1)^{|\alpha|}}{\pi^{n/4}\sqrt{2^{|\alpha|}\alpha!}} e^{x^2/2} \left(\frac{\partial}{\partial x}\right)^\alpha e^{-x^2}.$$

A polynomial given by

$$\begin{aligned} H_\alpha(x) &= \pi^{n/4}\sqrt{2^{|\alpha|}\alpha!} e^{x^2/2} h_\alpha(x) \\ &= (-1)^{|\alpha|} e^{x^2} \left(\frac{\partial}{\partial x}\right)^\alpha e^{-x^2} \\ &= \sum_{\beta \leqq [\alpha/2]} (-1)^{|\beta|} \frac{\alpha!}{\beta!(\alpha-2\beta)!} (2x)^{\alpha-2\beta} \end{aligned}$$

is said to be the α-th Hermite polynomial, which is a polynomial of x of order $|\alpha|$, where $[\alpha/2] = ([\alpha_1/2], \ldots, [\alpha_n/2])$ and $[t]$ is the largest integer less than or equat to $t \in \mathbb{R}$.

$h_\alpha(x)$ is am eigen function of the quantized Hamiltonian of harminic oscillator

$$H := P^*P + 1 = -\left(\frac{\partial}{\partial x}\right)^2 + x^2 = \sum_{j=1}^n \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right).$$

Theorem 6. *$Hh_\alpha = (2|\alpha|+1)h_\alpha$.*

Proof. Set $\tilde{h}_\alpha := \sqrt{2^{|\alpha|}\alpha!} h_\alpha = (P^*)^\alpha h_0$. Using Theorem —reftheorem:relation we deduce that

$$\begin{aligned} H\tilde{h}_\alpha &= \mathcal{B}^* \circ \mathcal{B} \circ P^*P(P^*)^\alpha h_0 + \tilde{h}_\alpha \\ &= \mathcal{B}^* \circ A^*A(A^*)^\alpha \varphi_0 + \tilde{h}_\alpha \\ &= \sum_{j=1}^n \mathcal{B}^* \circ A_j^* A_j (A^*)^\alpha \varphi_0 + \tilde{h}_\alpha \\ &= \sum_{j=1}^n \mathcal{B}^* \left(2z_j \frac{\partial}{\partial z_j} z^\alpha\right) + \tilde{h}_\alpha \\ &= \sum_{j=1}^n \mathcal{B}^* \left(2\alpha_j z^\alpha\right) + \tilde{h}_\alpha \\ &= 2|\alpha| \mathcal{B}^* \circ (A^*)^\alpha \varphi_0 + \tilde{h}_\alpha \\ &= 2|\alpha| (P^*)^\alpha \circ \mathcal{B}^* \varphi_0 + \tilde{h}_\alpha \\ &= (2|\alpha|+1)\tilde{h}_\alpha. \end{aligned}$$

This completes the proof. □

It is worth to mentioning that Simon developed the Fourier series

$$f = \sum_{\alpha \in \mathbb{N}_0^n} \langle f, h_\alpha \rangle_{L^2(\mathbb{R}^n)} h_\alpha, \quad f \in L^2(\mathbb{R}^n)$$

and gave the characterization of tempered distributions. See [7].

3 Weyl quantization on the Segal-Bargmann space

Following [8] we review the Weyl quantizations for $\mathcal{B}\big(\mathscr{S}(\mathbb{R}^n)\big)$. The Bargmann transform $\mathcal{B}$ is a global Fourier integral operator on $\mathbb{R}^n$ with a comples phase function $\phi(z,x)=i(z^2/4-zx+x^2/2)$, and is associated with a canonical transform giveb by

$$\kappa_{\mathcal{B}}:\mathbb{C}^n\times\mathbb{C}^n\ni(x,\xi)\mapsto\kappa_{\mathcal{B}}(x,\xi)=\left(x-i\xi,\frac{x+i\xi}{2i}\right)\mathbb{C}^n\times\mathbb{C}^n.$$

If we set $\theta(z):=\bar{z}/2i$ and $\Lambda:=\kappa_{\mathcal{B}}(\mathbb{R}^n\times\mathbb{R}^n)$, then $\Lambda=\{(z,\theta z):z\in\mathbb{C}^n\}$. In this section we introduce the Weyl quantization on Λ. Fix arbitrary $z\in\mathbb{C}^n$ and set

$$\Gamma(z)=\left\{\left(\zeta,\theta\left(\frac{z+\zeta}{2}\right)\right):\zeta\in\mathbb{C}^n\right\}.$$

The volume form of $\Gamma(z)$ is defined by

$$d\Omega:=d\zeta_1\wedge\cdots\wedge d\zeta_n\wedge d\theta_1\wedge\cdots\wedge d\theta_n,\quad(\zeta,\theta)\in\Gamma(z).$$

The reproducing formula (3) for $\mathcal{F}_n$ has an alternative representation of the form

$$U(z)=\frac{1}{(2\pi)^n}\int_{\Gamma(z)}e^{(z-\zeta)\theta}U(\zeta)d\Omega,\quad U\in\mathcal{F}_n. \tag{4}$$

This is similar to the Fourier inversion formula

$$u(x)=\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{i(x-y)\xi}u(y)dyd\xi,\quad u\in\mathscr{S}(\mathbb{R}^n).$$

We now introduce the Weyl quantization on $\mathbb{R}^n$ and on Λ. We denote by $\mathscr{A}(\mathbb{R}^n\times\mathbb{R}^n)$ the set of all smooth functions $a(x,\xi)$ on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying the following conditions: there exists a constant $m\in\mathbb{R}$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial\xi}\right)^\alpha a(x,\xi)\right|\leqq C(\alpha,\beta)(1+|x|+|\xi|)^m,\quad(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$$

for any $\alpha,\beta\in\mathbb{N}_0^n$ with some positive constant $C(\alpha,\beta)$. The Weyl quantization $\operatorname{Op}^{\mathrm{W}}(a)$ of $a(x,\xi)\in\mathscr{A}(\mathbb{R}^n\times\mathbb{R}^n)$ is defined by

$$\operatorname{Op}^{\mathrm{W}}(a)u(x):=\frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u(y)dyd\xi,\quad u\in\mathscr{S}(\mathbb{R}^n).$$

The symbol class $\mathscr{A}(\Lambda)$ on Λ is defined by

$$\mathscr{A}(\Lambda):=\{a:\Lambda\ni(z,\theta)\mapsto a(z,\theta)\in\mathbb{C},a\circ\kappa_{\mathcal{B}}\in\mathscr{A}(\mathbb{R}^n\times\mathbb{R}^n)\}.$$

The Weyl quantization $\operatorname{Op}^{\mathrm{W}}(a)$ of $a(z,\theta)\in\mathscr{A}(\Lambda)$ is defined by

$$\operatorname{Op}^{\mathrm{W}}(a)U(z):=\frac{1}{(2\pi)^n}\int_{\Gamma(z)}e^{i(z-\zeta)\theta}a\left(\frac{z+\zeta}{2},\theta\right)U(\zeta)d\Omega,\quad U\in\mathcal{B}\big(\mathscr{S}(\mathbb{R}^n)\big).$$

Using the Weyl quantization, we have

$$P=2i\operatorname{Op}^{\mathrm{W}}\left(\frac{x+i\xi}{2i}\right),\quad P^*=\operatorname{Op}^{\mathrm{W}}(x-i\xi),$$
$$A=2i\operatorname{Op}^{\mathrm{W}}(\theta),\qquad A^*=\operatorname{Op}^{\mathrm{W}}(z).$$

So we can express Theorem 4 as

$$\mathrm{Op}^{\mathrm{W}}\big((z,\theta)\big)\circ\mathcal{B}u=\mathcal{B}\circ\mathrm{Op}^{\mathrm{W}}\big(\kappa_{\mathcal{B}}(x,\xi)\big)u,\quad u\in\mathscr{S}(\mathbb{R}^n).$$

This formula can be generalized for more general symbols. It is possible to say that the Weyl quantization on $\mathbb{R}^n\times\mathbb{R}^n$ is equivalent to that on Λ via the Bargmann transform $\mathcal{B}$.

Theorem 7 (The Egorov theorem). *Suppose that $a(z,\theta)\in\mathscr{A}(\Lambda)$. Then*

$$\mathrm{Op}^{W}(a)\circ\mathcal{B}u=\mathcal{B}\circ\mathrm{Op}^{W}(a\circ\kappa_{\mathcal{B}})u,\quad u\in\mathscr{S}(\mathbb{R}^n).$$

Proof. The proof is basically elementary computation. Consider the distribution kernel of $\mathrm{Op}^{\mathrm{W}}(a)\circ\mathcal{B}$ and change the variables by the canonical transform $\kappa_{\mathcal{B}}$. We omit the detail. □

4 Berezin-Toeplitz operators on the Segal-Bargmann space

In this section we introduce Berezin-Toeplitz operators on $\mathcal{F}_n$. Let $b(z,\bar{z})$ be an appropriate function of $z\in\mathbb{C}^n$. The Berezin-Toeplitz operator T_b with a symbol b is defined by

$$T_bU(z):=\mathcal{B}\circ\mathcal{B}^*(bU)(z)=\frac{1}{(2\pi)^n}\int_{\mathbb{C}^n}e^{z\bar{\zeta}/2}b(\zeta,\bar{\zeta})U(\zeta)e^{-|\zeta|^2/2}L(d\zeta),\quad U\in\mathcal{F}_n.$$

We show that Berezin-Toeplitz operators are Weyl quantizations. Unfortunately, however, it is not known wheter Weyl quantizations are Berezin-Toeplitz operators.

Consider the initial value problem for a heat equation of the form

$$v_t-\Delta v=0,\quad v(0,z,\bar{z})=b(z,\bar{z}),\quad (t,z)\in(0,\infty)\times\mathbb{C}^n,$$

$$\Delta=2\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial\bar{z}}=2\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j\partial\bar{z}_j}.$$

The solution of this initial value problem is given by

$$e^{t\Delta}b(z,\bar{z})=\frac{1}{(2\pi t)^n}\int_{\mathbb{C}^n}e^{-|z-\zeta|^2/2t}b(\zeta,\bar{\zeta})L(d\zeta).$$

In [5] Guillemin discovered that Berezin-Toeplitz operators are Weyl quantizations on Λ.

Theorem 8. *If we set $a(z,\theta):=(e^{\Delta/2}b)(z,2i\theta)$, then $T_b=\mathrm{Op}(a)$.*
If we set $p(x,\xi):=(e^{\Delta/2}b)(x-i\xi,x+i\xi)$, then Theorem 7 implies that $T_b\circ\mathcal{B}=\mathcal{B}\circ\mathrm{Op}^{W}(p)$.

Let $b(z,\bar{z})$ be a bounded function on $\mathbb{C}^n$. For any $U,V\in\mathcal{F}_n$ we have

$$\langle T_bU,V\rangle_{\mathcal{F}_n}=\langle bU,V\rangle_{L^2(\mathbb{C}^n,e^{-|z|^2/2}L(dz))}$$

since $bU-T_bU\in\mathcal{F}_n^{\perp}$. Let E be a small neoghborhood of $x_0-i\xi_0$ in $\mathbb{C}^n$, and let χ_E be the characteristic function of E. Then $E':=\{(x,\xi):x-i\xi\in E\}$ is a small neighborhood of (x_0,ξ_0) in $\mathbb{R}^n\times\mathbb{R}^n$. In [2] Daubechies studied $\chi_E(z)\mathcal{B}u(z)$, which is the microlocalization of $u\in\mathscr{S}'(\mathbb{R}^n)$ near (x_0,ξ_0). What she had in mind was an application to time-frequency analysis. Daubechies' localization operator D_E is defined by

$$\langle D_Eu,v\rangle_{L^2(\mathbb{R}^n)}=\langle\chi_E\mathcal{B}u,\mathcal{B}v\rangle_{L^2(\mathbb{C}^n,e^{-|z|^2/2}L(dz))},\quad u,v\in\mathscr{S}(\mathbb{R}^n).$$

Then we have

$$\langle D_E u, v\rangle_{L^2(\mathbb{R}^n)} = \langle T_{\chi_E}\mathcal{B}u, \mathcal{B}v\rangle_{\mathcal{F}_n}, \quad u, v \in \mathscr{S}(\mathbb{R}^n).$$

So Theorem 8 implies that $D_E = \mathrm{Op}^{\mathrm{W}}(P_E)$ on $\mathbb{R}^n$, where

$$p_E(x,\xi) := e^{\Delta/2}\chi_E(x - i\xi, x + i\xi) = \frac{1}{\pi^n}\iint_{E'} e^{-(x-y)^2-(\xi-\eta)^2}\,dyd\eta.$$

This means that the Daubechies' localization operator D_E is the anti-Wick quantization of $\chi'_E(x,\xi)$.

Consider the rapidly decreasing function $e^{-(z^2/4-zx+x^2)}$ of x with a parameter z arising in the definition of the Bargmann transform. We can replace this by more general rapidly decreasing functions of x. Such functions are said to be window functions and the corresponding transforms are said to be windowed Fourier transform or short-time Fourier transform. Such transforms are studied mathematically and frequently used for communication theory, signal processing, image processing and etc. See [4].

5 FBI transform and Mexican hat wavelet

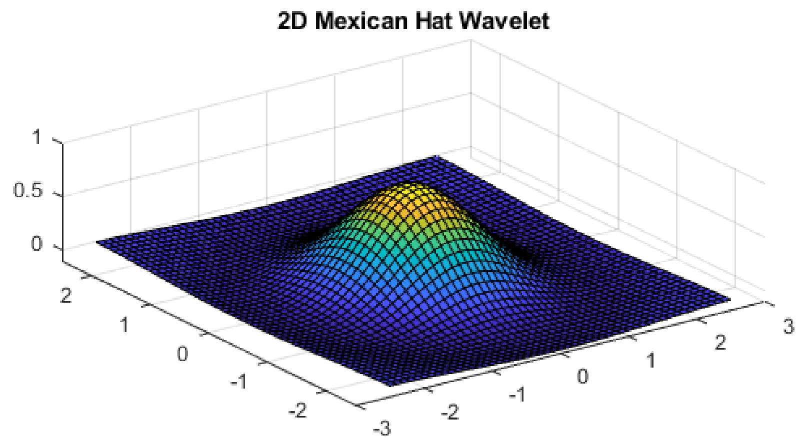
Let $\psi(x) \in \mathscr{S}(\mathbb{R}^n)$ be a wavelet function. See, e.g., [4]. The wavelet transform of $u \in \mathscr{S}'(\mathbb{R}^n)$ with the wavelet function ψ is defined by

$$W_\psi u(x,a) := a^{-n/2}\int_{\mathbb{R}^n} \overline{\psi\left(\frac{y-x}{a}\right)}u(y)dy, \quad x \in \mathbb{R}^n, a > 0.$$

The Mexican hat wavelet is defined by

$$\psi_{\mathrm{MEX}}(x) := -C\Delta_{\mathbb{R}^n}e^{-x^2/2} = -C\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}e^{-x^2/2} = \frac{2}{\pi^{n/4}\sqrt{n(n+2)}}(n - x^2)e^{-x^2/2},$$

where $C = 2/\pi^{n/4}\sqrt{n(n+2)}$ is chosen so that $\|\psi_{\mathrm{MEX}}\|_{L^2(\mathbb{R}^n)} = 1$. We denote by W_{MEX} the wavelet transform with the wavelet function ψ_{MEX}.



Here we introduce the semiclassical parameter $h > 0$. This is a very small parameter and corresponds to the reduced Plank constant $\hbar$. We define the semiclassical Bargmann transform and the FBI (Fourier-Bros-Iagornitzer) transform by

$$\begin{aligned}\mathcal{B}_h u(z) &= 2^{-n/2}(\pi h)^{-3n/4}\int_{\mathbb{R}^n} e^{-(z^2/4-zx+x^2/2)/h}u(y)dy,\\ \mathcal{T}_h u(x,\xi) &= e^{-|x-i\xi|^2/2h}\mathcal{B}_h u(x-i\xi)\\ &= 2^{-n/2}(\pi h)^{-3n/4}\int_{\mathbb{R}^n} e^{-i(x-y)\xi/h-(x-y)^2/2h}u(y)dy.\end{aligned}$$

Then we deduce that

$$\begin{aligned}W_{\text{MEX}}u(x,a) &= -Ca^{-n/2}\int_{\mathbb{R}^n}(a^2\Delta_{\mathbb{R}^n}e^{-(y-x)^2/2a^2})u(y)dy\\ &= Ca^{-n/2+2}\int_{\mathbb{R}^n} e^{-(y-x)^2/2a^2}\{-\Delta_{\mathbb{R}^n}u(y)\}dy\\ &= \frac{2}{\pi^{n/4}\sqrt{n(n+2)}}a^{-n/2+2}2^{n/2}(\pi a^2)^{3n/4}\mathcal{T}_{a^2}[-\Delta_{\mathbb{R}^n}u](x,a^2)\\ &= \frac{2^{n/2+1}\pi^{n/2}}{\sqrt{n(n+2)}}a^{n+2}\mathcal{T}_{a^2}[-\Delta_{\mathbb{R}^n}u](x,a^2).\end{aligned}$$

If $u \in \mathscr{S}'(\mathbb{R}^n)$ is independent of $h > 0$, then its analytic wave front set $\text{WF}_\text{A}(u)$, Gevrey-s wave front set $\text{WF}_s(u)$ ($s > 1$), and wave front set $\text{WF}(u)$ are respectively characterized as

$$\begin{aligned}(x_0,\xi_0)\notin \text{WF}_\text{A}(u) \quad &\Leftrightarrow \quad \mathcal{T}_h u(x,\xi) = \mathcal{O}(e^{-\delta/h}) \quad \text{near} \quad (x_0,\xi_0) \quad \text{as} \quad h\downarrow 0,\\ (x_0,\xi_0)\notin \text{WF}_s(u) \quad &\Leftrightarrow \quad \mathcal{T}_h u(x,\xi) = \mathcal{O}(e^{-\delta/h^{1/s}}) \quad \text{near} \quad (x_0,\xi_0) \quad \text{as} \quad h\downarrow 0,\\ (x_0,\xi_0)\notin \text{WF}(u) \quad &\Leftrightarrow \quad \mathcal{T}_h u(x,\xi) = \mathcal{O}(h^\infty) \quad \text{near} \quad (x_0,\xi_0) \quad \text{as} \quad h\downarrow 0.\end{aligned}$$

See [**?**, Martinez] Note that $e^{\xi^2/h}\mathcal{T}_h u(x,\xi)$ is holomorphic in $x-i\xi \in \mathbb{C}^n$. Considering Cauchy's integral formula, we deduce that $\mathcal{T}_h u(x,0)$ detects the singular support of u. The Laplacian $\Delta_{\mathbb{R}^n}$ is elliptic and does not affect the singular support. Thew we have the characterization of singular support via W_{MEX}.

Theorem 9.

$$\begin{aligned}x_0\notin \text{singsupp}_A(u) \quad &\Leftrightarrow \quad W_{MEX}u(x,a) = \mathcal{O}(e^{-\delta/a^2}) \quad \textit{near} \quad x_0 \quad \textit{as} \quad a\downarrow 0,\\ x_0\notin \text{singsupp}_s(u) \quad &\Leftrightarrow \quad W_{MEX}u(x,a) = \mathcal{O}(e^{-\delta/a^{2/s}}) \quad \textit{near} \quad x_0 \quad \textit{as} \quad a\downarrow 0,\\ x_0\notin \text{singsupp}(u) \quad &\Leftrightarrow \quad W_{MEX}u(x,a) = \mathcal{O}(a^\infty) \quad \textit{near} \quad x_0 \quad \textit{as} \quad a\downarrow 0.\end{aligned}$$

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