

# A NOTE ON D'ALEMBERT'S FORMULA

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ABSTRACT. We obtain d'Alembert's formula of solutions to the Cauchy problem for one-dimensional wave equation.

Consider the Cauchy problem for the one-dimensional wave equation of the form

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad \text{in } \mathbb{R}^2, \quad (1)$$

$$u(x, 0) = \varphi(x) \quad \text{in } \mathbb{R}, \quad (2)$$

$$u_t(x, 0) = \psi(x) \quad \text{in } \mathbb{R}, \quad (3)$$

where  $u(x, t)$  is an unknown function of  $(x, t) \in \mathbb{R}^2$ , and  $c > 0$ ,  $f(x, t) \in C(\mathbb{R}^{\neq})$ ,  $\varphi(x) \in C^2(\mathbb{R})$  and  $\psi(x) \in C^1(\mathbb{R})$  are given. The following theorem is well-known.

**Theorem 1.** *The Cauchy problem (1)-(2)-(3) has a unique solution  $u(x, t) \in C^2(\mathbb{R}^{\neq})$ , which is given by d'Alembert's formula*

$$u(x, t) = \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy \right) ds. \quad (4)$$

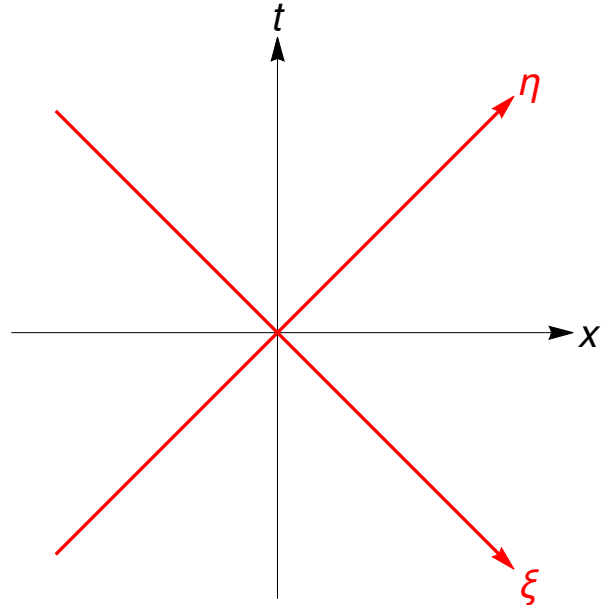
In what follows we shall obtain only d'Alembert formula (4). For this purpose we make use of a change of variables

$$(\xi, \eta) = (x - ct, x + ct),$$

that is,

$$(x, t) = \left( \frac{\xi + \eta}{2}, \frac{-\xi + \eta}{2c} \right).$$

The following figure describes the relationship between  $xt$ -plane and  $\xi\eta$ -plane. Note that  $x$ -axis ( $t = 0$ ) becomes the diagonal  $\{(\xi, \xi) \mid \xi \in \mathbb{R}\}$  in  $\xi\eta$ -plane.



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**Lemma 2.** Let  $u(x, t) \in C^2(\mathbb{R}^2)$  be a solution to (1)-(2)-(3). If we set

$$U(x - ct, x + ct) := u(x, t), \quad F(x - ct, x + ct) := f(x, t),$$

that is,

$$U(\xi, \eta) := u\left(\frac{\xi + \eta}{2}, \frac{-\xi + \eta}{2c}\right), \quad F(\xi, \eta) := f\left(\frac{\xi + \eta}{2}, \frac{-\xi + \eta}{2c}\right),$$

then  $U(\xi, \eta) \in C^2(\mathbb{R}^2)$  solves the Cauchy problem of the form

$$U_{\xi\eta}(\xi, \eta) = -\frac{F(\xi, \eta)}{4c^2} \quad \text{in } \mathbb{R}^2, \quad (5)$$

$$U(\xi, \xi) = \varphi(\xi) \quad \text{in } \mathbb{R}, \quad (6)$$

$$U_\eta(\xi, \xi) = \frac{\varphi'(\xi)}{2} + \frac{\psi(\xi)}{2c} \quad \text{in } \mathbb{R}. \quad (7)$$

*Proof.* Since  $u(x, t) = U(x - ct, x + ct)$ , we deduce that

$$\begin{aligned} u_t &= -cU_\xi + cU_\eta, \\ u_{tt} &= -c(-cU_\xi + cU_\eta)_\xi + c(-cU_\xi + cU_\eta)_\eta = c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}, \\ u_x &= U_\xi + U_\eta, \\ u_{xx} &= (U_\xi + U_\eta)_\xi + (U_\xi + U_\eta)_\eta = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}, \\ F &= f = u_{tt} - c^2u_{xx} = \{c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}\} - c^2\{U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\} = -4c^2U_{\xi\eta}, \\ \varphi(\xi) &= u(\xi, 0) = U(\xi, \xi), \\ \varphi'(\xi) &= U_\xi(\xi, \xi) + U_\eta(\xi, \xi), \\ \psi(\xi) &= u_t(\xi, 0) = -cU_\xi(\xi, \xi) + cU_\eta(\xi, \xi), \\ U_\eta(\xi, \xi) &= \frac{U_\xi(\xi, \xi) + U_\eta(\xi, \xi)}{2} + \frac{-U_\xi(\xi, \xi) + U_\eta(\xi, \xi)}{2} = \frac{\varphi'(\xi)}{2} + \frac{\psi(\xi)}{2c}. \end{aligned}$$

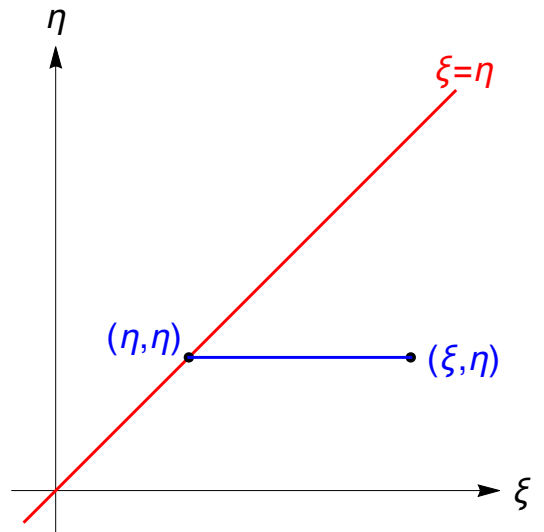
This completes the proof.  $\square$

We shall obtain d'Alembert's formula (4) by solving the Cauchy problem (5)-(6)-(7).

*Derivation of (4).* Suppose that  $u(x, t) \in C^2(\mathbb{R}^2)$  is a solution to (1)-(2)-(3). Lemma 2 shows that  $U \in C^2(\mathbb{R}^2)$  is a solution to the Cauchy problem (5)-(6)-(7).

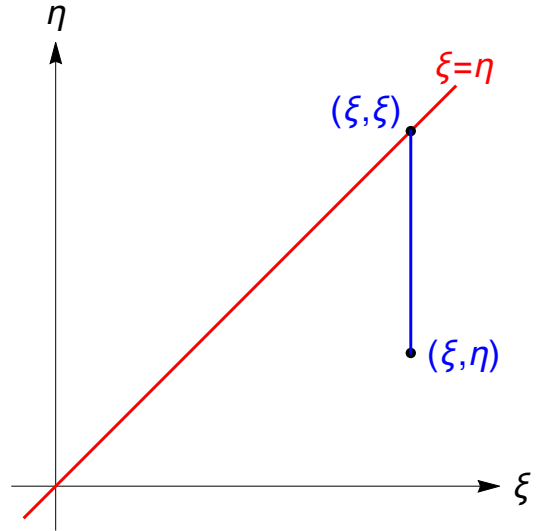
Firstly we integrate the equation (5) in  $\xi$  from  $\eta$  to  $\xi$ . See the next figure. We have

$$\begin{aligned} U_\eta(\xi, \eta) &= U_\eta(\eta, \eta) - \int_\eta^\xi \frac{F(z, \eta)}{4c^2} dz \\ &= \frac{\varphi'(\eta)}{2} + \frac{\psi(\eta)}{2c} + \int_\xi^\eta \frac{F(z, \eta)}{4c^2} dz. \end{aligned} \quad (8)$$



Secondly we integrate (8) in  $\eta$  from  $\xi$  to  $\eta$ . See the next figure. We have

$$\begin{aligned}
 U(\xi, \eta) &= U(\xi, \xi) + \int_{\xi}^{\eta} \left\{ \frac{\varphi'(y)}{2} + \frac{\psi(y)}{2c} \right\} dy \\
 &+ \int_{\xi}^{\eta} \left( \int_{\xi}^{\zeta} \frac{F(z, \zeta)}{4c^2} dz \right) d\zeta \\
 &= \varphi(\xi) + \int_{\xi}^{\eta} \left\{ \frac{\varphi'(y)}{2} + \frac{\psi(y)}{2c} \right\} dy \\
 &+ \int_{\xi}^{\eta} \left( \int_{\xi}^{\zeta} \frac{F(z, \zeta)}{4c^2} dz \right) d\zeta \\
 &= \frac{\varphi(\eta) + \varphi(\xi)}{2} + \frac{1}{2c} \int_{\xi}^{\eta} \psi(y) dy \\
 &+ \frac{1}{4c^2} \int_{\xi}^{\eta} \left( \int_{\xi}^{\zeta} F(z, \zeta) dz \right) d\zeta. \quad (9)
 \end{aligned}$$



Here we change variables by  $(\xi, \eta) = (x - ct, x + ct)$  and  $(z, \zeta) = (y - cs, y + cs)$  in (9). We have

$$\frac{\partial(z, \zeta)}{\partial(y, s)} = \det \begin{bmatrix} 1 & -c \\ 1 & c \end{bmatrix} = 2c.$$

In case of  $t > 0$  we have

$$\xi = x - ct < \zeta = y + cs < \eta = x + ct, \quad \xi = x - ct < z = y - cs < \zeta = y + cs < \eta.$$

Hence  $0 \leq s \leq t$  and  $x - c(t - s) \leq y \leq x + c(t - s)$ . Similarly, in case of  $t < 0$  we have  $t \leq s \leq 0$  and  $x + c(t - s) \leq y \leq x - c(t - s)$ . Then we have

$$\begin{aligned}
 u(x, t) &= U(x - ct, x + ct) \\
 &= \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{4c^2} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \cdot (2c) dy \right) ds \\
 &= \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + \frac{1}{2c} \int_0^t \left( \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy \right) ds.
 \end{aligned}$$

This completes the proof. □