# Geometric Analysis of Dispersive Flows

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# Contents

1. Model equations arising in classical physics
2. Schrödinger flow into almost Hermitian manifolds
3. Third order flow into almost Hermitian manifolds
4. Fourth order flow into Kähher manifolds

# 1-1. Model equations arising in classical physics

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx}, \tag{1}$$

$$\vec{u} = {}^t(u_1, u_2, u_3) : \mathbb{R} \times \mathbb{R} \ni (t, x) \mapsto \vec{u}(t, x) \in \mathbb{S}^2.$$

- $\vec{u}$ describes not only a point in the unit 2-sphere $\mathbb{S}^2$, but also the unit normal vector of the tangential plane of $\mathbb{S}^2$ at $\vec{u}$.
- $\vec{u} \times \vec{u}_{xx} = \vec{u} \times (\vec{u}_{xx})^T$ where ,

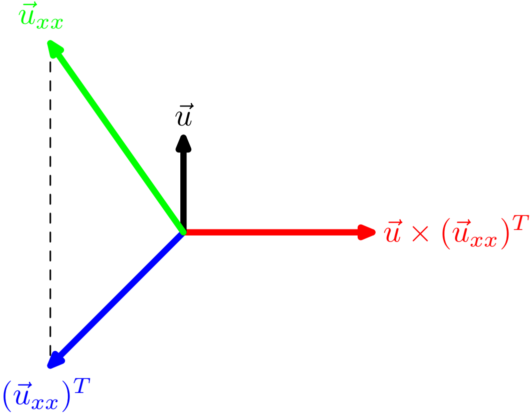
$$(\vec{u}_{xx})^T := \vec{u}_{xx} - \langle \vec{u}_{xx}, \vec{u} \rangle \vec{u} \in T_{\vec{u}(t,x)}\mathbb{S}^2$$

  is the tangential component of $\vec{u}_{xx}$ at $\vec{u}$. $\vec{u} \perp (\vec{u}_{xx})^T$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^3$.
- $\vec{u} \perp \vec{u} \times (\vec{u}_{xx})^T$ and $(\vec{u}_{xx})^T \perp \vec{u} \times (\vec{u}_{xx})^T$ since

$$\langle \vec{a}, \vec{b} \times \vec{c} \rangle = \det[\vec{a}, \vec{b}, \vec{c}], \quad \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3.$$

# 1-2. The meaning of $\vec{u} \times \vec{u}_{xx}$

- $\{\vec{u}, (\vec{u}_{xx})^T, \vec{u} \times (\vec{u}_{xx})^T\}$ is a right-handed orthogonal system of $\mathbb{R}^3$ if $(\vec{u}_{xx})^T \neq \vec{0}$:

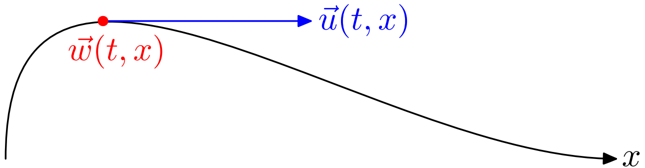



- If $(\vec{u}_{xx})^T$ rotates **90**-degree in the anti-clockwise direction on $T_{\vec{u}(t,x)}\mathbb{S}^2$, then it becomes $\vec{u} \times \vec{u}_{xx}$, that is $\vec{u} \times (\vec{u}_{xx})^T$.

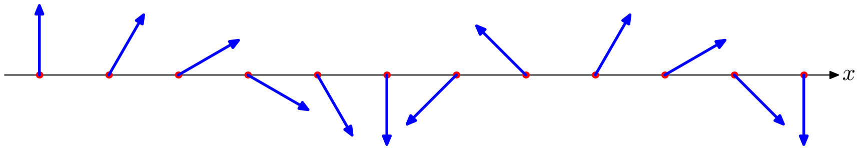
# 1-3. What does $\vec{u}_t = \vec{u} \times \vec{u}_{xx}$ model?

The system (1) models the following phenomena in classical physics:

- The motion of vortex filament (Da Rios, 1906),
  Let $\vec{w}(t, \cdot)$ be a curve in $\mathbb{R}^3$ at the time $t \in \mathbb{R}$ parametrized by the arclength $x \in \mathbb{R}$. Set $\vec{u} = \vec{w}_x$.




- Heisenberg's ferromagnetic model (continium limit of classical spin chain system).



# 1-4. The structure of the system of PDEs $\vec{u}_t = \vec{u} \times \vec{u}_{xx}$

If $\vec{u}(t,x)$ never takes a value of a fixed point, e.g., the north pole ${}^t(0,0,1) \in \mathbb{S}^2$, then the system (1) becomes a semilinear Schrödinger equation of the form

$$v_t - \sqrt{-1}v_{xx} = \frac{2\sqrt{-1}\bar{v}(v_x)^2}{1+|v|^2} \tag{2}$$

where, $v$ is the stereographic projection of $\vec{u}$ defined by

$$\mathbb{S}^2 \ni \vec{u} = (u_1, u_2, u_3) \mapsto v = \frac{u_1 + \sqrt{-1}u_2}{1-u_3} \in \mathbb{C}.$$

The solutions to (2) may be uninteresting in physics. In particular, small solutions to (2) are definitely uninteresting since those are very close to the constant map $\vec{u}(t,x) \equiv {}^t(0,0,1)$.

# 1-5. Higher order models

- A third order model

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx} + a \left[ \vec{u}_{xxx} + \frac{3}{2} \{ \vec{u}_x \times (\vec{u} \times \vec{u}_x) \}_x \right], \qquad (3)$$

  where $a \in \mathbb{R} \setminus \{0\}$, was proposed by Fukumoto and Miyazaki (1991) for the motion of vortex filaments.

- A fourth order model

$$\vec{u}_t = \vec{u} \times \left\{ a\vec{u}_{xxxx} + \vec{u}_{xx} + b\langle \vec{u}_x, \vec{u}_x \rangle \vec{u}_{xx} + c\langle \vec{u}_{xx}, \vec{u}_x \rangle \vec{u}_x \right\}, \qquad (4)$$

  where $a, b, c \in \mathbb{R}$ satisfy $3a + c = 2b$ and $a \neq 0$, was proposed by Porsezian, Daniel and Lakshmanan (1992) for spin chain system and by Fukumoto (2002) for the motion of vortex filaments respectively.

- If all the constsns are zero, then the model equations (3) and (4) become (1) respectively.

# 1-6. Geometric generalization of model systems

Model systems (1), (3) and (4) can be geometrically generalized as follows

$$u_t = J_u \nabla_x u_x, \tag{5}$$

$$u_t = J_u \nabla_x u_x + a\nabla_x^2 u_x + b h_u(u_x, u_x) u_x, \tag{6}$$

$$\begin{aligned} u_t = {} & J_u \nabla_x u_x + a J_u \nabla_x^3 u_x \\ & + b h_u(u_x, u_x) J_u \nabla_x u_x + c h_u(\nabla_x u_x, u_x) J_u u_x, \end{aligned} \tag{7}$$

where $a, b, c \in \mathbb{R}$ are constants, $u : \mathbb{R} \times \mathbb{R} \to N$, $(N, J, h)$ is a $2n$-dimensional compact almost Hermitian manifold, $\nabla_x$ is the covariant derivative along the mapping $u$, which is roughly speaking

$$\nabla_x u_x = \text{ the projecton of } u_{xx} \text{ onto } T_{u(t,x)} N.$$

(5), (6) and (7) are identities in $T_{u(t,x)} N$ for each $(t, x) \in \mathbb{R} \times M$. The purpose of the geometric generalization is to understand the structure of the systems essentially.

# 1-7. Target manifold $(N, J, h)$

We assume that $(N, J, h)$ is a compact almost Hermitian manifold, which is the least setting on $(N, J, h)$ so that the identities (5), (6) and (7) make sense. More precisely we assume the following

- $J$ is an almost complex structure, that is, $J_u$ is a linear transformation and $J_u \circ J_u = -I_d$ on $T_u N$ for all $u \in N$.
- $h$ is a Riemannian metric satisfying $h(J\cdot, J\cdot) = h(\cdot, \cdot)$.

$(N, J, h)$ is said to be a Kähler manifold if

- $J$ is integrable, that is, $(N, J)$ is a complex manifold.
- The Kähler form $h(J\cdot, \cdot)$ is cloed, that is, $d\big(h(J\cdot, \cdot)\big) = 0$.

It is well-known that these conditions are equivalent to

$$\nabla^N J = 0, \tag{8}$$

where $\nabla^N$ is the Levi-Civita connection of $(N, J, h)$. We sometimes assume that $(N, J, h)$ is a compact Kähler manifold. In this case, (5) and other equations become easy to handle.

# 2-1. Schrödinger flow into almost Hermitian manifolds

The equation (5) can be more generalized as follows:

$$u_t = J_u \tau(u) \quad \text{in} \quad \mathbb{R} \times M \tag{9}$$

where $u : \mathbb{R} \times M \to N$, $(M, g)$ is an $m$-dimensional closed Riemannian manifold, $(N, J, h)$ is a $2n$-dimensional compact almost Hermitian manifold, $\tau(u)$ is the tension field of the mapping $u(t, \cdot) : M \to N$, which is roughly speaking

$$\tau(u)(t, x) = \text{the projection of } \Delta_g u(t, x) \text{ onto } T_{u(t,x)} N.$$

Solutions to (9) is said to be Schrödinger maps.

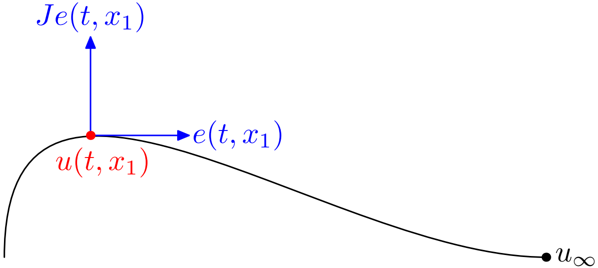
- Stationary solutions to (9) satisfy $\tau(u) = 0$ and are said to be harmonic maps.
- If $M = \mathbb{S}^1 \simeq \mathbb{T} := \mathbb{R}/\mathbb{Z}$, then the solutions to (9) describes the motion of closed curves on $N$.

We study the IVP for (9) with the initial condition of the form

$$u(0, x) = u_0(x) \quad \text{in} \quad M. \tag{10}$$

# 2-2. Geometric reduction of Schrödinger flows

- Chang-Shatah-Uhlenbeck (CPAM, 2000)
  $M = \mathbb{R}$, $(N, J, h)$ is a compact Riemann surface, and
  $\exists u_\infty \in N$ s.t. $u(t,x) \to u_\infty$ as $x \to \infty$ for $\forall t \in \mathbb{R}$.
  $\Rightarrow \exists$ a moving frame along the curve $u(t,\cdot)$.




- Nahmod-Shatah-Vega-Zeng (IMRN, 2008)
  $M = \mathbb{R}^m$, $\nabla^N J = 0 \Rightarrow \exists$ a moving frame along the map $u(t,\cdot)$.
- The Kähler condition $\nabla^N J = 0$ plays an essential role in the construction of the moving frames: $\nabla_x e = 0 \Leftrightarrow \nabla_x Je = 0$.

# 2-3. Analysis of the IVPs

- Sulem-Sulem-Bardos (CMP, 1985)
  Short-time existence and time-global weak solutions for (1).
- Koiso (Osaka J. Math., 1997)   $M = \mathbb{S}^1$ and $\nabla^N J = 0$.
  1. Reformulation of (1) as (9).
  2. Short-time existence.
  3. $\nabla^N R = 0 \Rightarrow$ Time-global existence.
     $R(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z$
     for $X, Y, Z \in \mathcal{X}(N)$.
- The Kähler condition $\nabla^N J = 0$ guarantees the classical energy estimates for (9).
- C (BLMS, 2013): Short-time existence without $\nabla^N J = 0$.

# 2-4. Short-time existence 1/3

## Theorem 1 (C (BLMS, 2013))

*Let* $k \in \mathbb{N}$ *satisfying* $2k > m/2 + 5$, *and let* $k_0 = \min k$. *Then,* $\forall u_0 \in H^{2k}(M;TN)$, $\exists T = T(\|u\|_{H^{2k_0}}) > 0$, *such that* (9)-(10) *has a unique solution* $u \in C([-T,T]; H^{2k}(M;TN))$.

Remarks

- Set $G = \det[g_{ij}]$ and $\nabla_i = \nabla^N_{du(\partial/\partial x^i)}$. Then,

  $$\tilde{\Delta}_g = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{m} \nabla_i g^{ij} \sqrt{G} \nabla_j \text{ is globally well-defined.}$$

- $$\|u\|^2_{H^{2k}(M;TN)} \sim \sum_{l=1}^{k} \int_M h(\tilde{\Delta}^l_g u, \tilde{\Delta}^l_g u)\, d\mu_g, \quad \tilde{\Delta}^l_g u = \tilde{\Delta}^{l-1}_g \tau(u).$$

- $u \in H^{2k_0}(M;TN) \Rightarrow \nabla_i u \in C^4(M;TN)$.

  Our calculus requires this smoothness of $\nabla_i u$.

# 2-4. Short-time existence 2/3

We show the idea of proof. Set $v = \tilde{\Delta}_g^{k-1}\tau(u)$. Then,

$$\left\{ \nabla_t - \frac{1}{\sqrt{G}} \sum_{i,j=1}^{m} \nabla_i g^{ij} \sqrt{G} J_u \nabla_j - P \right\} v = \text{harmless terms}, \qquad (11)$$

where

$$P = (2k-1) \sum_{i,j=1}^{m} g^{ij} \{\nabla_i J_u\} \nabla_j$$

is an anti-symmetric first-order term, and an obstruction to establishing short-time existence theorem. To eliminate this, we introduce a PsDO $\Lambda = 1 - \tilde{\Lambda}$ acting on $\Gamma(u^{-1}TN)$ as follows. $\tilde{\Lambda}$ is of order $-1$. Rougly speaking,

$$\tilde{\Lambda} = \frac{PJ_u(1-\Delta_g)^{-1}}{2} \simeq -\frac{J_u P(1-\Delta_g)^{-1}}{2}$$

locally. Fortunately, this expression is invariant under the change of variables of $M$ and $N$, and makes sense globally.

# 2-4. Short-time existence 3/3

Set

$$\tilde{\Lambda} = \sum_{\alpha} \sum_{\beta} \phi_{\alpha}(x) \psi_{\beta}(u) \tilde{\Lambda}_{\alpha,\beta} \Phi_{\alpha}(x) \Psi_{\beta}(u),$$

where

$$\phi_{\alpha}, \Phi_{\alpha} \in C_0^{\infty}(M), \quad \sum_{\alpha} \phi_{\alpha} = 1, \quad \Phi_{\alpha} = 1 \text{ in } \operatorname{supp}[\phi_{\alpha}],$$

$$\psi_{\beta}, \Psi_{\beta} \in C_0^{\infty}(N), \quad \sum_{\beta} \psi_{\beta} = 1, \quad \Psi_{\beta} = 1 \text{ in } \operatorname{supp}[\psi_{\beta}],$$

and $\tilde{\Lambda}_{\alpha,\beta}$ is a local expression of $\tilde{\Lambda}$. Then, $\Lambda$ works well and

$$\left\{ \nabla_t - \frac{1}{\sqrt{G}} \sum_{i,j=1}^{m} \nabla_i g^{ij} \sqrt{G} J_u \nabla_j \right\} \Lambda v = \text{harmless terms.}$$

# 3. Third-order flows for closed curves

We study the IVP for (6) of the form

$$u_t = a\nabla_x^2 u_x + J_u \nabla_x u_x + b h_u(u_x, u_x) u_x \quad \text{in} \quad \mathbb{R}\times\mathbb{T}, \tag{12}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}, \tag{13}$$

where $a, b \in \mathbb{R}$ and $a \neq 0$. Known results are the following

- Onodera (SIGMA, 2008): Geometric reduction in case that $(N, J, h)$ is a compact Riemann surface.
- Onodera (JGA, 2008):
  - Short-time existence for $\nabla^N J = 0$.
  - Time-global existence for the case that $(N, J, h)$ is a compact Riemann surface with constant curvature $K$ and $b = aK/2$.
- C-Onodera (JFA, 2009): Short-time existence without $\nabla^N J = 0$.
- Onodera (CommPDE, 2010): Time-global existence for the case that $(N, J, h)$ is a Kähler manifold with $\nabla^N R = 0$.

# 4-1. Fourth-order flows for closed curves

We study the IVP for (6) of the form

$$\begin{aligned} u_t &= aJ_u\nabla_x^3 u_x + \{1 + bh_u(u_x, u_x)\}J_u\nabla_x u_x \\ &\quad + ch_u(\nabla_x u_x, u_x)J_u u_x. && \text{in} \quad \mathbb{R}\times\mathbb{T}, \quad (14) \\ u(0,x) &= u_0(x) && \text{in} \quad \mathbb{T}, \quad (15) \end{aligned}$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Known results are the following

- Onodera (SIGMA, 2008): Geometric reduction.
- C (J. PsDOs Appl., 2015):
  - Geometric reduction for higher-order derivatives.
  - It is almost impossible to construct a solution to the IVP (14)-(15) unless the sectional curvature $K(u)$ of $N$ is constant.
- Onodera (Edinburgh, 2017; JGA, 2017; ...): Short-time existence for 2-sphere, Riemann surface with constant curvature and more...
- C-Onodera (Z. Anal. Anwend., 2015): c.f. short-time existence for $x \in \mathbb{R}$ and $\nabla^N J = 0$.

# 4-2. IVP for a linear dispersive system 1/2

$$\left\{I_2\frac{\partial}{\partial t}+J_2\frac{\partial^4}{\partial x^4}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\frac{\partial}{\partial x}\right\}\vec{u}=\vec{f}(t,x), \qquad (16)$$

$$\vec{u}(0,x)=\vec{u}_0(x), \qquad (17)$$

where

- $\mathbb{R}\times\mathbb{T}\ni(t,x)\mapsto\vec{u}(t,x)\in\mathbb{R}^2$ is unknown.
- $I_2$, $J_2$, $B(x)$ and $C(x)$ are $2\times 2$ real matrices:

$$I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad J_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad B(x), C(x)\in C^\infty\bigl(\mathbb{T};M_2(\mathbb{R})\bigr).$$

Note that if

$$B(x)+{}^tB(x)=0, \quad C(x)-{}^tC(x)=0,$$

then the classical energy estimates work for (9).

# 4-2. IVP for a linear dispersive system 2/2

**Theorem 2 (C (2015))**

*The following conditions are mutually equivalent.*

(I) *For any* $\vec{u}_0 \in L^2(\mathbb{T};\mathbb{R}^2)$ *and for any* $\vec{f} \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{T};\mathbb{R}^2))$*, the IVP* (16)-(17) *has a unique solution* $\vec{u} \in C(\mathbb{R}; L^2(\mathbb{T};\mathbb{R}^2))$*.*

(II) $\int_0^1 \operatorname{tr}(B(x))\,dx = 0$ *and* $\int_0^1 \operatorname{tr}(J_2 C(x))\,dx = 0$*.*

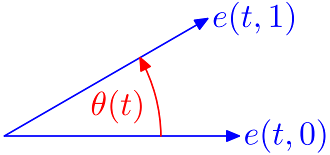
- $\operatorname{tr}(B(x)) = b_{11}(x) + b_{22}(x)$, $\operatorname{tr}(J_2 C(x)) = c_{12}(x) - c_{21}(x)$.
- (I) implies the continuity of $(\vec{u}_0, \vec{f}) \mapsto \vec{u}$ automatically.
- Proof.
  Consider more general systems. By using PsDOs, we can diagonalize (9) essentially. The proof is reduced to Mizuhara's results on single equations (2006).

# 4-3. Geometric reduction of dispersive flows 1/3

- Let $u \in C^\infty(\mathbb{R} \times \mathbb{T}; N)$ be a solution to (14).
  Suppose that $u_x(t,0) \neq 0$ for all $t \in \mathbb{R}$. Set

$$e_0(t) = \frac{u_x(t,0)}{\sqrt{g_{u(t,0)}(u_x, u_x)}}, \quad \gamma(t) = \{u(t,x) \mid x \in \mathbb{T}\}.$$

- Let $e(t,x)$ be a solution to $\nabla_x e = 0$, $e(t,0) = e_0(t)$.
  $\{e, Je\}$ is a moving frame along $\gamma(t)$.
  $\nabla_x Je = 0$, $Je(t,0) = Je_0(t)$ since $\nabla^N J = 0$.




  $e(t,0) \neq e(t,1)$ in general.
- Let $\theta(t)$ be the holonomy angle of $\gamma(t)$.
  If $\gamma(t)$ is the boundary of a contractive domain $\Omega(t)$, then

$$\theta(t) = \int_{\Omega(t)} K(u)\, du^1 \wedge du^2.$$

# 4-3. Geometric reduction of dispersive flows 2/3

- Set $\nabla_x^\ell u_x = Ve + WJe$ for $\ell = 4, 5, 6, \dots$.
- Let $R$ be the Riemann curvature tensor of $(N, J, h)$.
- Using the following properties

$$K(u) = h_u\big(R(Je, e)e, Je\big), \quad JR(X, Y)Z = R(X, Y)JZ,$$

$$R(Je, e)e = K(u)Je, \quad JR(Je, e)e = -K(u)e, \dots,$$

we obtain

$$\left\{ I_2 \frac{\partial}{\partial t} - aJ_2 \frac{\partial^4}{\partial x^4} + \hat{B}(t, x) \frac{\partial^2}{\partial x^2} + \hat{C}(t, x) \frac{\partial}{\partial x} \right\} \begin{bmatrix} V \\ W \end{bmatrix} = \cdots,$$

- Unfortunately, $V(t, x)$ and $W(t, x)$ are not necessarily periodic in $x$ because of the holonomy angle $\theta(t)$.

# 4-3. Geometric reduction of dispersive flows 3/3

- Correction

$$\vec{z} = \begin{bmatrix} \cos(\theta(t)x) & -\sin(\theta(t)x) \\ \sin(\theta(t)x) & \cos(\theta(t)x) \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}.$$

- $\vec{z} \in C^\infty(\mathbb{R} \times \mathbb{T}; \mathbb{R}^2)$ and $\vec{z}$ solves

$$\left\{ I_2 \frac{\partial}{\partial t} - aJ_2 \frac{\partial^4}{\partial x^4} - a\theta(t) I_2 \frac{\partial^3}{\partial x^3} + \hat{B}_1(t,x) \frac{\partial^2}{\partial x^2} + \hat{C}_1(t,x) \frac{\partial}{\partial x} \right\} \vec{z} = \cdots,$$

- Note that $\mathbf{tr}(\hat{B}_1(t,x)) \equiv 0$ and

$$\int_0^1 \mathbf{tr}(J_2 \hat{C}_1(t,x))\,dx = -\frac{a}{2} \int_0^1 h_u(u_x, u_x) \frac{\partial}{\partial x} K(u)\,dx.$$

We cannot expect the LHS to vanish in general unless $K(u)$ is constant.

# Thank you for your attention.