# A NOTE ON EULER KERNELS AND ONE-DIMENSIONAL RIESZ POTENTIALS

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ABSTRACT. We compute some Fourier transformations to understand the relationship between Euler kernels and one-dimensional Riesz potentials.

For a complex number $\lambda$ satisfying $\operatorname{Re}(\lambda) > 0$, set

$$H^{\lambda}_{\pm}(x) = \frac{(\pm x)^{\lambda-1}_{+}}{\Gamma(\lambda)}, \quad x \in \mathbb{R},$$

where $t_+ = t$ for $t > 0$ and $t_+ = 0$ for $t \leqslant 0$, and $\Gamma(\lambda)$ is the gamma function defined by

$$\Gamma(\lambda) := \int_0^\infty e^{-t} t^{\lambda-1} dt, \quad \operatorname{Re}(\lambda) > 0.$$

Inpatricular, $H^1(x)$ is so-called the Heaviside function $Y(x)$. $H^{\lambda}_{+}$ and $H^{\lambda}_{-}$ are said to be Euler kernels. We denote by $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ the Schwartz class, which is the set of all rapidily decreasing functions, and its topological dual, which is the set of all tempered distributions, on $\mathbb{R}$ respectively. It is easy to see that $H^{\lambda}_{\pm} \in \mathscr{S}'(\mathbb{R})$ for $\operatorname{Re}(\lambda) > 0$, and we can consider its Fourier transform. For $f \in \mathscr{S}(\mathbb{R})$, its Fourier transform and its inverse Fourier transform are defined by

$$\mathscr{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R},$$

$$\mathscr{F}^*[f](x) := \hat{f}(-x) = \int_{\mathbb{R}} e^{2\pi i x \xi} f(\xi) d\xi, \quad x \in \mathbb{R}.$$

**Proposition 1.** *We have for* $\operatorname{Re}\lambda > 0$,

$$\mathscr{F}[H^{\lambda}_{\pm}](\xi) = (\pm 2\pi i \xi + 0)^{-\lambda} := \lim_{\varepsilon \downarrow 0} (\pm 2\pi i \xi + \varepsilon)^{-\lambda}.$$

*Proof.* It suffices to show $\mathscr{F}[H^{\lambda}_{+}](\xi) = (2\pi i \xi + 0)^{-\lambda}$ since $H^{\lambda}_{-}(x) =^{\lambda}_{+} (-x)$. Since

$$\int_{\mathbb{R}} |H^{\lambda}_{+}(x)| e^{-\varepsilon x} dx < \infty, \qquad H^{\lambda}_{+}(x) e^{-\varepsilon x} \to H^{\lambda}_{+}(x) \quad \text{in} \quad \mathscr{S}'(\mathbb{R}) \quad (\varepsilon \downarrow 0),$$

We have

$$\mathscr{F}[H^{\lambda}_{+}](\xi) = \lim_{\varepsilon \downarrow 0} \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(2\pi i \xi + \varepsilon)x} x^{\lambda-1} dx.$$

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By using Cauchy's theorem for $e^{-\zeta}\zeta^{\lambda-1}$ in the half plane $\mathrm{Re}(\zeta) > 0$, we deduce that

$$\begin{aligned}\mathscr{F}[H_+^\lambda(\cdot)e^{-\varepsilon x}](\xi) &= \frac{(2\pi i\xi+\varepsilon)^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty e^{-(2\pi i\xi+\varepsilon)x}\{(2\pi i\xi+\varepsilon)x\}^{\lambda-1}(2\pi i\xi+\varepsilon)dx \\ &= (2\pi i\xi+\varepsilon)^{-\lambda}.\end{aligned}$$

Hence we have $\mathscr{F}[H_+^\lambda](\xi) = (2\pi i\xi+0)^\lambda$. □

Here we inrofuce one-dimensional Riesz potential $I_\lambda(x)$ for $0 < \mathrm{Re}(\lambda) < 1$. This is defined by

$$I_\lambda(x) := \mathscr{F}^*\left[\frac{1}{(2\pi|\xi|)^\lambda}\right](x).$$

It is well-known the folowing facts.

**Proposition 2.** *We have for* $0 < \mathrm{Re}(\lambda) < 1$

$$I_\lambda(x) = C_\lambda|x|^{\lambda-1}, \quad C_\lambda = 2^{-\lambda}\pi^{-1/2}\Gamma\left(\frac{1-\lambda}{2}\right)\Big/\Gamma\left(\frac{\lambda}{2}\right).$$

*Proof.* It is easy to see that $I_\lambda(-x) = I_\lambda(x)$ and $I_\lambda(tx) = t^{\lambda-1}I_\lambda x$ for $t > 0$. Thus, if we set $C_\lambda := I_\lambda(1)$, then $I_\lambda(x) = C_\lambda|x|^{\lambda-1}$. It is not easy to compute $I_\lambda(1)$. We here use Fourier transform of tempered distributions. Recall $\mathscr{F}^*[e^{-\pi x^2}](\xi) = e^{-\pi\xi^2}$. Then have

$$\begin{aligned}\int_{-\infty}^\infty \frac{C_\lambda}{|x|^{1-\lambda}}e^{-\pi x^2}dx &= \left\langle \frac{C_\lambda}{|x|^{1-\lambda}}, e^{-\pi x^2}\right\rangle \\ &= \left\langle \frac{1}{(2\pi|x|)^\lambda}, e^{-\pi\xi^2}\right\rangle = \int_{-\infty}^\infty \frac{1}{(2\pi|x|)^\lambda}e^{-\pi\xi^2}d\xi.\end{aligned}$$

By using the change of variable $\pi x^2 = s$ and $\pi\xi^2 = t$ respectively, we obtain

$$\begin{aligned}\int_{-\infty}^\infty \frac{C_\lambda}{|x|^{1-\lambda}}e^{-\pi x^2}dx 2\int_0^\infty \frac{C_\lambda}{|x|^{1-\lambda}}e^{-\pi x^2}dx& \\ &= C_\lambda\pi^{-\lambda/2}\int_0^\infty e^{-s}s^{\lambda/2-1}ds \\ &= C_\lambda\pi^{-\lambda/2}\Gamma\left(\frac{\lambda}{2}\right), \\ \int_{-\infty}^\infty \frac{1}{(2\pi|x|)^\lambda}e^{-\pi\xi^2}d\xi = 2\int_0^\infty \frac{1}{(2\pi|x|)^\lambda}e^{-\pi\xi^2}d\xi& \\ &= 2^{-\lambda}\pi^{-(\lambda+1)/2}\int_0^\infty e^{-t}t^{(1-\lambda)/2-1}dt \\ &= 2^{-\lambda}\pi^{-(\lambda+1)/2}\Gamma\left(\frac{1-\lambda}{2}\right).\end{aligned}$$

Comparering them, we obtain

$$C_\lambda = 2^{-\lambda}\pi^{-1/2}\Gamma\left(\frac{1-\lambda}{2}\right)\Big/\Gamma\left(\frac{\lambda}{2}\right).$$

This completes the proof. □

For $t \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$, we deduce that

$$\begin{aligned}(it+\varepsilon)^{-\lambda} &= e^{-\lambda \operatorname{Log}(it+\varepsilon)} = e^{-\lambda\left\{\log\sqrt{t^2+\varepsilon^2}+i\operatorname{Arg}(it+\varepsilon)\right\}} \\ &= e^{-\lambda\left\{\log|t|+i\pi\operatorname{sgn}(t)/2+o(1)\right\}} = \frac{e^{-i\pi\lambda\operatorname{sgn}(t)/2}}{|t|^\lambda}\left\{1+o(1)\right\}, \\ (it+0)^{-\lambda} &= \frac{e^{-i\pi\lambda\operatorname{sgn}(t)/2}}{|t|^\lambda}.\end{aligned}$$

By using this, we have the following.

**Proposition 3.** *We have for* $0 < \operatorname{Re}(\lambda) < 1$

$$(\pm 2\pi i\xi + 0)^{-\lambda} = \frac{e^{\mp i\pi\lambda\operatorname{sgn}(\xi)/2}}{(2\pi|\xi|)^\lambda}, \quad \textit{i.e.,} \quad \mathscr{F}[H_\pm^\lambda](\xi) = \mathscr{F}[I_\lambda](\xi)e^{\mp i\pi\lambda\operatorname{sgn}(\xi)/2},$$

By using the Euler kernels and the Riesz potentials, we can consider some kind of integration of order $\lambda$ for $0 < \operatorname{Re}(\lambda) < 1$ as follows. For $f \in \mathscr{S}(\mathbb{R})$, we define

$$\begin{aligned}F_\lambda(x) &:= H_+^\lambda * f(x) = \frac{1}{\Gamma(\lambda)}\int_{-\infty}^x (x-y)^{\lambda-1}f(y)dy, \\ G_\lambda(x) &:= I_\lambda * f(x) = C_\lambda\int_{-\infty}^\infty \frac{f(y)}{|x-y|^{1-\lambda}}dy.\end{aligned}$$

Note that

$$F_1(x) = \int_{-\infty}^x f(y)dy$$

is a primitive function of $f$.

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