# X-RAY TRANSFORM ON THE PLANE AND RECONSTRUCTION FORMULA: THE PRINCIPLES OF COMPUTED TOMOGRAPHY

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ABSTRACT. This note reviews the basic facts on Fourier analysis on the Euclidean space in brief and introduce X-ray transform on the plane and reconstruction formula, which are the principles of computed tomography (CT).

## 1. FOURIER TRANSFORM ON THE EUCLIDEAN SPACE

In this section we recall the Fourier transform on the Euclidean space and summarize its basic facts needed to construct the theory on the X-ray transform on the plane. Denote the set of all nonnegative integers by $\mathbb{N}_0$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and multi-indeces $\alpha = (\alpha_1, \dots, \alpha_n) and \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, we set

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}, \quad \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We say that $f(x)$ is a rapidly decreasing funtion on $\mathbb{R}^n$ if $f(x) \in C^\infty(\mathbb{R}^n)$ and

$$\sup_{x \in \mathbb{R}^n} \left| x^\beta \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| < \infty$$

for any $\alpha, \beta \in \mathbb{N}_0^n$. The set of all rapidly deceasing funtions on $\mathbb{R}^n$ is denoted by $\mathscr{S}(\mathbb{R}^n)$. Note that if $f(x) \in \mathscr{S}(\mathbb{R}^n)$ then $x^\beta \partial_x^\alpha f(x) \in \mathscr{S}(\mathbb{R}^n)$ and $f(x)$ is integrable on $\mathbb{R}^n$ since $f(x) = \mathcal{O}\big((1 + |x|)^{-(n+1)}\big)$ as $|x| \to \infty$.

**Definition 1.** Let $f(x)$ be an integrable function on $\mathbb{R}^n$. The Fourier transform $\mathcal{F}f(\xi) = \hat{f}(\xi)$ of $f(x)$ is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n,$$

where $x \cdot \xi$ is the standard inner product on $\mathbb{R}^n$. The inverse Fourier transform of $g$ is also defined by $\mathcal{F}^* g(x) := \mathcal{F}g(-x)$.

In what follows we introduce basic facts on Fourier transform. Most of them are stated as the properties of rapidly decreasing funtions, but hold for functions of a wider class. These can be justfied since tempered distributions can be approximated by sequences of rapidly decreasing functions in some sense. In this note we omit the detail. Firstly we state the elementary facts on Fourier transform by simple computation.

**Theorem 2.** *Assume that $f, g \in \mathscr{S}(\mathbb{R}^n)$, $h \in \mathbb{R}^n$.*

- $\mathcal{F}[e^{-2\pi i x \cdot h} f](\xi) = \hat{f}(\xi + h), \quad \mathcal{F}[f(\cdot + h)](\xi) = e^{2\pi i h \cdot \xi} \hat{f}(\xi).$
- $\mathcal{F}\left[\dfrac{\partial f}{\partial x_j}\right](\xi) = (2\pi i \xi_j) \cdot \hat{f}(\xi), \quad \mathcal{F}[(2\pi i x_j) \cdot f](\xi) = -\dfrac{\partial \hat{f}}{\partial \xi_j}(\xi).$
- $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ *is a linear mapping.*
- $\mathcal{F}[f * g](\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$, *where $f * g$ is said to be the comvolution of $f$ and $g$ defined by*

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy.$$

*Proof.* By simple computation, we have

$$\mathcal{F}[e^{-2\pi i x\cdot h}f](\xi) = \int_{\mathbb{R}^2} e^{-2\pi i x\cdot(\xi+h)}f(x)dx = \hat{f}(\xi+h),$$

$$\begin{aligned}\mathcal{F}[f(\cdot+h)](\xi) &= \int_{\mathbb{R}^2} e^{-2\pi i x\cdot\xi}f(x+h)dx \\ &= \int_{\mathbb{R}^2} e^{-2\pi i(y-h)\cdot\xi}f(y)dy = e^{2\pi i h\cdot\xi}\hat{f}(\xi),\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left[\frac{\partial f}{\partial x_j}\right](\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\frac{\partial f}{\partial x_j}(x)dx = -\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_j}e^{-2\pi i x\cdot\xi}\right)f(x)dx \\ &= (2\pi i\xi_j)\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}f(x)dx = (2\pi i\xi_j)\cdot\hat{f}(\xi),\end{aligned}$$

$$\begin{aligned}\mathcal{F}[(2\pi i x_j)\cdot f](\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\cdot(2\pi i x_j)\cdot f(x)dx \\ &= -\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial \xi_j}e^{-2\pi i x\cdot\xi}\right)f(x)dx \\ &= -\frac{\partial}{\partial\xi_j}\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}f(x)dx = -\frac{\partial\hat{f}}{\partial\xi_j}(\xi).\end{aligned}$$

Repeating this computation, we can obtain

$$(2\pi i\xi)^\alpha\frac{\partial^\beta\hat{f}}{\partial\xi^\beta}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\frac{\partial^\alpha}{\partial x^\alpha}\Big\{(2\pi i x)^\beta\cdot f(x)\Big\}dx. \tag{1}$$

Since $f(x)\in\mathscr{S}(\mathbb{R}^n)$, the integrand can be evaluated by

$$C(f,\alpha,\beta)(1+|x|)^{-n-1}$$

multiplied by some positive constant, where

$$C(f,\alpha,\beta) := \sum_{\substack{\alpha'\leqq\alpha \\ \beta'\leqq\beta}}\sup_{y\in\mathbb{R}^n}(1+|y|)^{n+1}\left|y^{\beta'}\frac{\partial^{\alpha'}f}{\partial y^{\alpha'}}(y)\right|.$$

Hence the left hand side of (1) is bounded. So it follows that $\mathcal{F}:\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$. By simple computation again, we deduce that

$$\begin{aligned}\mathcal{F}[f*g](\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\left(\int_{\mathbb{R}^n} f(x-y)g(y)dy\right)dx \\ &= \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}g(y)\left(\int_{\mathbb{R}^n} e^{-2\pi i(x-y)\cdot\xi}f(x-y)dx\right)dy \\ &= \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}g(y)\left(\int_{\mathbb{R}^n} e^{-2\pi i z\cdot\xi}f(z)dz\right)dy \\ &= \int_{\mathbb{R}^n} e^{-2\pi i z\cdot\xi}f(z)dz\cdot\int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi}g(y)dy \\ &= \hat{f}(\xi)\cdot\hat{g}(\xi).\end{aligned}$$

This completes the proof. □

The following result is the most important fact on the mapping properties of the Fourier transform.

**Theorem 3.** *For $f \in \mathscr{S}(\mathbb{R}^n)$ we have*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n. \tag{2}$$

*This is said to be the Fourier inversion formula. Moreover $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a linear bijection, and $\mathcal{F}^* \circ \mathcal{F} f = f$ and $\mathcal{F} \circ \mathcal{F}^* f = f$ hold.*

It suffices to prove the the Fourier inversion formula (2). The other part was proved in the proof of Theorem 2. We need the following two lemmas to prove (2).

**Lemma 4.** *We have $\mathcal{F}[e^{-\pi|x|^2}](\xi) = e^{-\pi|\xi|^2}$, where $|x|^2 = x\cdot x$.*

*Proof.* We have only to prove the case of $n = 1$ since $e^{-\pi|x|^2} = e^{-\pi(x_1^2+\cdots+x_n^2)}$. If we set $n = 1$ and $F(\xi) = \mathcal{F}[e^{-\pi x^2}](\xi)$, we have

$$\begin{aligned} F(0)^2 &= \iint_{\mathbb{R}^2} e^{-\pi(x^2+y^2)} dxdy = \int_0^{2\pi} \int_0^r e^{-\pi r^2} r dr d\theta \\ &= \int_0^\infty e^{-\pi r^2} 2\pi r dr = \left[ -\frac{d}{dr} e^{-\pi r^2} \right]_0^\infty = 1 \end{aligned}$$

and $F(0) = 1$. Moreover, we obtain

$$\begin{aligned} F'(\xi) + 2\pi\xi F(\xi) &= \int_{-\infty}^\infty (-2\pi i x + 2\pi\xi) e^{-2\pi i x\xi - \pi x^2} dx \\ &= i \int_{-\infty}^\infty \frac{\partial}{\partial x} e^{-2\pi i x\xi - \pi x^2} dx = 0. \end{aligned}$$

Thus $F(\xi)$ solves the initial value problem $(e^{\pi\xi^2} F(\xi))' = 0$, $F(0) = 1$, and we get $F(\xi) = e^{-\pi\xi^2}$. □

Next lemma is said to be the multiplication formula.

**Lemma 5.** *For $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x) dx = \int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi) d\xi.$$

This seems to be similar to the Plancherel-Parseval formula

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)} dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi.$$

Actually, however, the multiplication formula is completely defferent from the Plancherel-Parseval formula. The multiplication formula is independent of the inversion formula and hepls the proof of the inversion formula. The proof of the Plancherel-Parseval formula needs contrary the inversion formula.

**Proof of Lemma 5.** By simple computation we deduce that

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\hat{g}(x) dx &= \int_{\mathbb{R}^n} f(x) \left( \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} g(\xi) d\xi \right) dx \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x) dx \right) g(\xi) d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi) d\xi, \end{aligned}$$

which is desired. □

Here we prove Theorem 3.

**Proof of Theorem 3.** Suppose that $f \in \mathscr{S}(\mathbb{R}^n)$. Fix arbitrary $x \in \mathbb{R}^n$. Set $g_\delta(\xi) = e^{-\delta^2\pi|\xi|^2}$ with $\delta > 0$. Then we have

$$\widehat{g_\delta}(y) = \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\xi} e^{-\delta^2\pi|\xi|^2} d\xi = \delta^{-n} \int_{\mathbb{R}^n} e^{-2\pi i y\cdot\eta/\delta} e^{-\pi|\eta|^2} d\xi = \delta^{-n} e^{-\pi|y|^2/\delta^2}.$$

The Fourier transform of $f(x+y)$ in $y$ is $e^{2\pi i x\cdot\xi}\hat{f}(\xi)$. Applying the multiplication formula to $f(x+\cdot)$ and $g_\delta(\cdot)$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\pi|z|^2} f(x+\delta z)dz &= \delta^{-n} \int_{\mathbb{R}^n} f(x+y) e^{-\pi|y|^2/\delta^2} dy = \int_{\mathbb{R}^n} f(x+y)\widehat{g_\delta}(y)dy \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi) g_\delta(\xi)d\xi = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi) e^{-\delta^2\pi|\xi|^2} d\xi.\end{aligned}$$

Letting $\delta \downarrow 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\pi|z|^2} f(x+\delta z)dz &\to f(x)\cdot \int_{\mathbb{R}^n} e^{-\pi|z|^2}dz = f(x), \\ \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi) e^{-\delta^2\pi|\xi|^2} d\xi &\to \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi,\end{aligned}$$

and establish the inversion formula (2). □

Finally we obtain the Fourier transform of the principal value of $1/x$ which will be used in the proof of the reconstruction formula for the X-ray transform. In what follows we assume that $n = 1$. We begin with the computation of the Dirichelet integral.

**Lemma 6.**

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

*Proof.* This is an improper Riemann integral, and we consider

$$\begin{aligned}I(\varepsilon, R) &= \int_\varepsilon^R \frac{\sin x}{x}dx = \frac{1}{2i}\int_\varepsilon^R \frac{e^{ix}-e^{-ix}}{x}dx \\ &= \frac{1}{2i}\left\{\int_\varepsilon^R \frac{e^{-ix}}{-x}dx + \int_\varepsilon^R \frac{e^{ix}}{x}dx\right\} = \frac{1}{2i}\left\{\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}dx + \int_\varepsilon^R \frac{e^{ix}}{x}dx\right\}\end{aligned}$$

with $0 < \varepsilon < R$. By using the decomposition

$$\begin{aligned}I(\varepsilon, R) &= \int_\varepsilon^{2\pi} \frac{\sin x}{x}dx + \int_{2\pi}^{2(N+1)\pi} \frac{\sin x}{x}dx + \int_{2(N+1)\pi}^{R} \frac{\sin x}{x}dx \\ &= \int_\varepsilon^{2\pi} \frac{\sin x}{x}dx + \sum_{n=1}^N \int_{2n\pi}^{2(n+1)\pi} \frac{\sin x}{x}dx + \mathcal{O}(R^{-1}) \\ &= \int_\varepsilon^{2\pi} \frac{\sin x}{x}dx + \sum_{n=1}^N \int_0^\pi \frac{\pi\sin x}{(2n\pi+x)\bigl((2n+1)\pi+x\bigr)}dx + \mathcal{O}(R^{-1})\end{aligned}$$

for $0 < \varepsilon < 2\pi$ and $2(N+1)\pi \leqq R < 2(N+2)\partial$, and $\sin x/x \to 1$ as $x \downarrow 0$, we deduce that the existence of the Dirichelet integral, that is,

$$I(\varepsilon, R) \to \int_0^\infty \frac{\sin x}{x}dx \quad (\varepsilon \downarrow 0, R \to \infty). \tag{3}$$

Here we consider a holomorphic function $f(z) = e^{iz}/2iz$ on $\mathbb{C}\setminus\{0\}$. We integrate this on the contour of the form

$$[-R, -\varepsilon] \cup \bigl(-C(\varepsilon)\bigr) \cup [\varepsilon, R] \cup C(R), \quad C(\rho) = \{\rho e^{i\theta} \mid \theta \in [0,\pi]\}, \rho > 0.$$

Cauchy's integral theorem implies that

$$I(\varepsilon, R) = \int_{C(\varepsilon)} f(z)dz - \int_{C(R)} f(z)dz. \tag{4}$$

We look at the two terms in the right hand side of (4) respectively. On one hand the first term satisfies

$$\int_{C(\varepsilon)} f(z)dz = \frac{1}{2i}\int_0^\pi \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta} d\theta = \frac{1}{2}\int_0^\pi e^{i\varepsilon e^{i\theta}} d\theta \to \frac{\pi}{2} \quad (\varepsilon \downarrow 0). \tag{5}$$

On the other hand the second term is evaluated as

$$\begin{aligned}\left|\int_{C(R)} f(z)dz\right| &= \frac{1}{2}\left|\int_0^\pi e^{iRe^{i\theta}} d\theta\right| \leqq \frac{1}{2}\int_0^\pi |e^{iRe^{i\theta}}| d\theta = \frac{1}{2}\int_0^\pi e^{-R\sin\theta} d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} e^{-R\sin\theta} d\theta + \frac{1}{2}\int_{\pi/2}^\pi e^{-R\sin\theta} d\theta = \int_0^{\pi/2} e^{-R\sin\theta} d\theta,\end{aligned}$$

where we used the change of variable for $\theta \in [\pi/2, \pi]$ by $\theta' = \pi - \theta \in [0, \pi/2]$, and replace $\theta'$ by $\theta$. In view of (3), (4) and (5), we have only to show

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \to 0 \quad (R \to \infty) \tag{6}$$

to complete the proof of Lemma 6. For this purpose we study a function $g(\theta) = \sin\theta - 2\theta/\pi$, $\theta \in [0, \pi/2]$. Since

$$g(0) = g\left(\frac{\pi}{2}\right) = 0,$$

$$g'(\theta) = \cos\theta - \frac{2}{\pi}, \quad g'(0) = 1 - \frac{2}{\pi} > 0, \quad g'\left(\frac{\pi}{2}\right) = -\frac{2}{\pi} < 0$$

$$g''(0) = 0, \qquad g''(\theta) = -\sin\theta < 0 \quad (\theta \in (0, \pi/2]),$$

the derivative test chart of $g(\theta)$, $\theta \in [0, \pi/2]$ is as follows.

| $\theta$ | $0$ | | $\mathrm{Arccos}(2/\pi)$ | | $\pi/2$ |
|---|---|---|---|---|---|
| $g''(\theta)$ | $0$ | $-$ | $-$ | $-$ | $-$ |
| $g'(\theta)$ | $+$ | $+$ | $0$ | $-$ | $-$ |
| $g(\theta)$ | $0$ | $\nearrow$ | | $\searrow$ | $0$ |

This shows that if $\theta \in [0, \pi/2]$ then $g(\theta) \geqq 0$, and $\sin\theta \geqq 2\theta/\pi$. By using this fact we deduce that

$$0 \leqq \int_0^{\pi/2} e^{-R\sin\theta} d\theta \leqq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{2R}(1 - e^{-R}) \leqq \frac{\pi}{2R},$$

which implies that (6) holds. This completes the proof. □

Here we compute the Fourier transform of the principal value of $1/x$.

**Lemma 7.** *We have*

$$\mathcal{F}\left[\mathrm{vp}\,\frac{1}{t}\right](\tau) := \lim_{\varepsilon \downarrow 0} \int_{|t|>\varepsilon} \frac{e^{-2\pi i t\tau}}{t} dt = -i\pi\,\mathrm{sgn}(\tau),$$

*where* $\mathrm{sgn}(\tau) := \pm 1$ *for* $\pm\tau > 0$.

**証明.** Suppose that $\tau \neq 0$. Using the change of variable $x = 2\pi t\tau$ we have

$$\begin{aligned}\int_{|t|>\varepsilon} \frac{e^{-2\pi i t\tau}}{t} dt &= \int_{|t|>\varepsilon} \frac{\cos(2\pi t\tau) - i\sin(2\pi t\tau)}{t} dt \\ &= -2i\int_\varepsilon^\infty \frac{\sin(2\pi t\tau)}{t} dt\end{aligned}$$

$$
\begin{aligned}
&= -2i \int_{\varepsilon}^{\infty} \frac{\sin(2\pi t\tau)}{2\pi t\tau} \cdot (2\pi\tau) dt \\
&= \begin{cases} -2i \displaystyle\int_{2\pi|\tau|\varepsilon}^{\infty} \frac{\sin x}{x} dx & (\tau > 0) \\ 2i \displaystyle\int_{-\infty}^{-2\pi|\tau|\varepsilon} \frac{\sin x}{x} dx & (\tau < 0) \end{cases} \\
&= \begin{cases} -2i \displaystyle\int_{2\pi|\tau|\varepsilon}^{\infty} \frac{\sin x}{x} dx & (\tau > 0) \\ 2i \displaystyle\int_{2\pi|\tau|\varepsilon}^{\infty} \frac{\sin x}{x} dx & (\tau < 0) \end{cases} \\
&= -2i \operatorname{sgn}(\tau) \int_{2\pi|\tau|\varepsilon}^{\infty} \frac{\sin x}{x} dx \\
&\to -2i \operatorname{sgn}(\tau) \int_{0}^{\infty} \frac{\sin x}{x} dx = -i\pi \operatorname{sgn}(\tau) \quad (\varepsilon \downarrow 0).
\end{aligned}
$$

This completes the proof. □

## 2. X-RAY TRANSFORM ON THE PLANE AND RECONSTRUCTION FORMULA

In this section we introduce the X-ray transform of scalar functions on the plane, and obtain the reconstruction formula. We also explain the formulation of CT scanners and see that the X-ray transform can be interpreted as the observation of CT scanners.

We begin with the definition of the X-ray transform. Roughly speaking, the X-ray transform is the collection of the integration of a function on all lines in the plane, that is, the X-ray transform maps a function on the plane to a function of lines on the plane. Here we introduce a parametrization of planar lines. Let $(\theta, t) \in [0, 2\pi] \times \mathbb{R}$. A planar line of the direction $(-\sin\theta, \cos\theta)$ is described by the direction and a point through which the line passes. Such a point is uniquely determined if we choose the nearest one from the origine, and is given by $(t\cos\theta, t\sin\theta)$ with some $t \in \mathbb{R}$. So we can deal with all the planar lines of the form

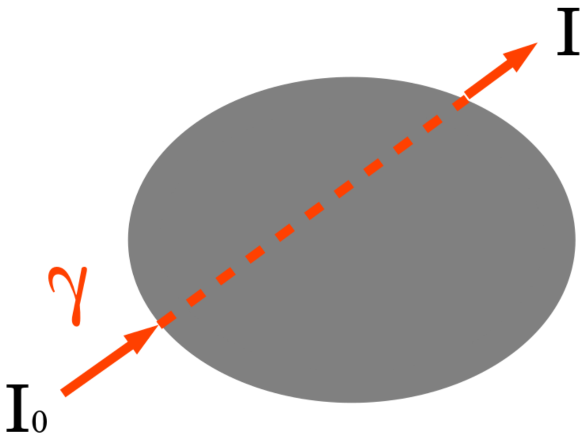
$$
\begin{aligned}
L(\theta, t) &= \{(x, y) \in \mathbb{R}^2 \mid x\cos\theta + y\sin\theta = t\} \\
&= \{(t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta) \mid \sigma \in \mathbb{R}\}.
\end{aligned}
$$

Note that $L(\theta \pm \pi, -t) = L(\theta, t)$. The X-ray transform of an appropriate function $f(x, y)$ on the plane is defined by

$$
\mathcal{X}f(\theta, t) := \int_{-\infty}^{\infty} f(t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta) d\sigma.
$$

Note that $\mathcal{X}f(\theta \pm \pi, -t) = \mathcal{X}f(\theta, t)$ holds.

Next we explain the formulation of CT scanners. Here we suppose that $f(x, y)$ is a density function of a cross section of media. We explain the X-ray transform of $f$ can be regarded as the observation of the computer tomography. For each cross section of media, a CT scanner irradiates the section with various X-rays along with lines on the section, and observe the intensity of X-rays after passing through bodies to produce cross-sectional images. CT scanners are mainly used for medical images. These equipments enable us to see the inside by using the information from the surfaces. Firstly we consider the case that $f(x, y)$ is an isotorpic media with uniform density $d > 0$ over a bounded domain $\Omega$, that is, $f(x, y) = d$ for $(x, y) \in \Omega$, and $f(x, y) = 0$ for $(x, y) \notin \Omega$.

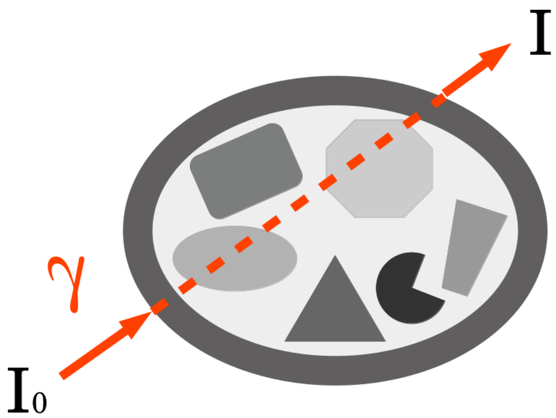



If one irradiate the bounded domain with X-ray of intensity $I_0$ along a line $\gamma$, the X-ray travels the length $\ell$ in $\Omega$ and passes through $\Omega$ with intensity $I$. We can intuitively understand that the absorption of X-ray is propotional to the density $d$ and the travel length $\ell$. Lambert's law of absorption in optics states that $-\log(I/I_0) > 0$ is proportional to $d\ell$. If we here suppose that raleted physical constant is normalized, then we have $-\log(I/I_0) = d\ell$. This is interpreted as

$$-\log\left(\frac{I}{I_0}\right) = \int_\gamma f = \mathcal{X}f(\gamma), \quad \text{i.e.,} \quad \frac{I}{I_0} = \exp\left(-\int_\gamma f\right) = \exp\Big(-\mathcal{X}f(\gamma)\Big). \tag{7}$$

The left hand side of (7) is given by the observation of CT scanner. So the right hand side and the X-ray transform also can be regarded as the observation of CT scanner.

Next we consider the case that $f(x, y)$ is a step function supported in $\Omega$. In other words, we study a body consisting of finite numbers of isotropic media.




The restriction of $f(x, y)$ on a line $\gamma$ is a one-variable step function on $\gamma$. Repeating (7) for isotropic media, we can see that (7) holds also. Indeed if we denote the restriction of $f$ on $\gamma$ by $g(\sigma) := f|_\gamma$, $\sigma \in \mathbb{R}$, and $\sigma[a, b]$ corresponds to the part in $\Omega$, we have

$$g(\sigma) = \sum_{k=1}^{n} d_k \chi_{[a_{k-1},a_k]}(\sigma), \quad a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b, \quad d_1, \ldots, d_n > 0,$$

where $\chi_{[a_{k-1},a_k]}(\sigma)$ the characteristic function of the interval $[a_{k-1}, a_k]$, that is, $\chi_{[a_{k-1},a_k]}(\sigma) = 1$ for $\sigma \in [a_{k-1}, a_k]$, and $\chi_{[a_{k-1},a_k]}(\sigma) = 0$ otherwise. On $\gamma$, the media is a joint segment of $n$ isotropic segments. The part of $[a_{k=1}, a_k]$ is a media of the length $a_k - a_{k-1}$ and the density $d_k$. Suppose that X-ray enter the body at the point $\sigma = a$ with the intensity $I_0$, and quit the body at $\sigma = b$ with the intensity $I$. Denote the intensity at $\sigma = a_k$ by $I_k$. Then $I = I_n$, and for each isotropic media

$$-\log\left(\frac{I_k}{I_{k-1}}\right) = d_k(a_k - a_{k-1}), \quad k = 1, 2, 3, \ldots, n.$$

Summing up this on $k$ up to $n$, we have

$$-\log\left(\frac{I}{I_0}\right) = -\log\left(\frac{I_n}{I_{n-1}}\cdot\dots\cdot\frac{I_2}{I_1}\cdot\frac{I_1}{I_0}\right) = -\sum_{k=1}^{n}\log\left(\frac{I_k}{I_{k-1}}\right)$$
$$= \sum_{k=1}^{n} d_k(a_k - a_{k-1}) = \int_a^b g(\sigma)d\sigma = \int_{-\infty}^{\infty} g(\sigma)d\sigma = \int_\gamma f = \mathcal{X}f(\gamma).$$

Thus we obtain (7).

In the case of a general function $f(x,y)$ of an appropriate class, we approximate $f$ by a sequence of step funtions and take a limit. Since (7) holds for all step funtions, this holds also for $f(x,y)$. Hence $\mathcal{X}f$ can be interpreted as the observation of CT scanners for general functions $f(x,y)$.

Based on the argument of the formulation above, we believe that

Can $f$ be reconstructed by given $\mathcal{X}f$?

is an important problem not only in mathematics but also in various applied science and engineering including medical imaging, baggage inspection and etc. We see that this problem is afirmatively solved below.

The next theorem is said to be the Fourier slice theorem of X-ray transform, and guarantees the uniqueness of reconstruction. Indeed if the theorem holds, $Xf = 0$ implies that $\hat{f} = 0$ and then $f = 0$.

**Theorem 8.**
$$\hat{f}(\tau\cos\theta, \tau\sin\theta) = \int_{-\infty}^{\infty} e^{-2\pi i t\tau}\mathcal{X}f(\theta,t)dt, \quad \tau\in\mathbb{R}.$$

*Proof.* Note that the inner product on the plane

$$(t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta)\cdot(\tau\cos\theta, \tau\sin\theta) = t\tau,$$

and

$$(x,y) = (t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta)$$

shows that the change of variables $(t,\sigma)\mapsto(x,y)$ is a rotation on the plane. Hence we deduce that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-2\pi i t\tau}\mathcal{X}f(\theta,t)dt &= \iint_{\mathbb{R}^2} e^{-2\pi i t\tau} f(t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta)d\sigma dt\\ &= \iint_{\mathbb{R}^2} e^{-2\pi i(t\cos\theta-\sigma\sin\theta, t\sin\theta+\sigma\cos\theta)\cdot(\tau\cos\theta,\tau\sin\theta)}\\ &\qquad\times f(t\cos\theta - \sigma\sin\theta, t\sin\theta + \sigma\cos\theta)d\sigma dt\\ &= \iint_{\mathbb{R}^2} e^{-2\pi i(x\tau\cos\theta + y\tau\sin\theta)} f(x,y)dxdy\\ &= \hat{f}(\tau\cos\theta, \tau\sin\theta),\end{aligned}$$

which is desired. □

Finally we prove the reconstruction formula and conclude this note.

**Theorem 9.** *Assume that* $f(x,y) = O\big((1+|x|+|y|)^{-1-\delta}\big)$, $\delta > 0$. *Then we have*
$$f(x,y) = \frac{1}{2\pi^2}\int_0^\pi\left(\mathrm{vp}\int_{-\infty}^{\infty}\frac{\partial_t\mathcal{X}f(\theta,t)}{x\cos\theta + y\sin\theta - t}dt\right)d\theta. \tag{8}$$

*Proof.* The change of variables on the plane of the form

$$(\xi,\eta) = (\tau\cos\theta, \tau\sin\theta), \quad (\tau,\theta)\in\mathbb{R}\times[0,\pi]$$

is essentially same as the polar coordinates on the plane, the absolute value of its Jacobian is $|\tau|$. Set

$$\widehat{\mathcal{X}f}(\theta,\tau) = \int_{-\infty}^{\infty} e^{-2\pi i t\tau}\mathcal{X}f(\theta,t)dt.$$

Using

- The Fourier inversion formula in Theorem 3 for $\mathbb{R}^2$.
- The Fourier slice theorem in Theorem 8 for X-ray transform.
- Change of variables $(\xi,\eta) = (\tau\cos\theta, \tau\sin\theta)$.
- Application of the Fourier inversion of the Fourier transform of convolutions in Theorem 2 for $\mathbb{R}^1$
$$\int_{-\infty}^{\infty} f(\sigma-t)g(t)dt = \int_{-\infty}^{\infty} e^{2\pi i\sigma\tau}\hat{f}(\tau)\hat{g}(\tau)d\tau,$$
to the Fourier transform of the principal value of $1/x$ in Lemma 7.

in the order, we deduce that

$$\begin{aligned}
f(x,y) &= \iint_{\mathbb{R}^2} e^{2\pi i(x\xi+y\eta)}\hat{f}(\xi,\eta)d\xi d\eta \\
&= \int_0^\pi \int_{-\infty}^{\infty} e^{2\pi\tau(x\cos\theta+y\sin\theta)}\widehat{\mathcal{X}f}(\theta,\tau)|\tau|d\tau d\theta \\
&= \frac{1}{2\pi^2}\int_0^\pi \int_{-\infty}^{\infty} e^{2\pi\tau(x\cos\theta+y\sin\theta)}(-i\pi\operatorname{sgn}(\tau))\cdot(2\pi i\tau)\cdot\widehat{\mathcal{X}f}(\theta,\tau)d\tau d\theta \\
&= \frac{1}{2\pi^2}\int_0^\pi \int_{-\infty}^{\infty} e^{2\pi\tau(x\cos\theta+y\sin\theta)}(-i\pi\operatorname{sgn}(\tau))\cdot\widehat{\partial_t\mathcal{X}f}(\theta,\tau)d\tau d\theta \\
&= \frac{1}{2\pi^2}\int_0^\pi \left(\mathrm{vp}\int_{-\infty}^{\infty} \frac{\partial_t\mathcal{X}f(\theta,t)}{x\cos\theta+y\sin\theta-t}dt\right)d\theta.
\end{aligned}$$

This completes the proof. □

We remark that (8) can be expressed as

$$\begin{aligned}
f(x,y) &= \Lambda\circ\mathcal{X}^*\circ\mathcal{X}f(x,y), \\
\mathcal{X}^*F(x,y) &:= \frac{1}{4\pi}\int_0^{2\pi} F(\theta, x\cos\theta+y\sin\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{\pi} F(\theta, x\cos\theta+y\sin\theta)d\theta, \\
\Lambda f(x,y) &= \left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{1/2} f(x,y) \\
&:= \iint_{\mathbb{R}^2} e^{2\pi i(x\xi+y\eta)}\left(|2\pi\xi|^2+|2\pi\eta|^2\right)^{1/2}\hat{f}(\xi,\eta)d\xi d\eta,
\end{aligned}$$

where $(\theta,t)$ is a funtion satisfying $F(\theta\pm\pi,-t) = F(\theta,t)$. $\mathcal{X}^*$ is said to be the dual X-ray transform, and $\Lambda$ is said to be the square root of the Laplacian. In applied scieces $\mathcal{X}^*$ is called the (unfilterd) back projection, $\Lambda$ or its numerical realization is called the filter, and the reconstruction formula (8) or its numerical realization is called the filtered back projection.

That is all of this note.