Taylor's Theorem and Taylor Expansion

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Mean Value Theorem

Theorem 1

Let $[a, b]$ be a closed interval. Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

Equivalently there exists $\theta \in (0, 1)$ such that

$$f(b) = f(a) + f'\big(\theta b + (1 - \theta)a\big)(b - a).$$

Proof of Theorem 1 i

We define a function $F(x)$ by

$$F(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

Then $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover $F(a) = F(b) = 0$.
The derivative of $F(x)$ is

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

It suffices to show that there exists $c \in a, b$ such that $F'(c) = 0$.
If $F(x) \equiv 0$, then $F'(x) \equiv 0$ on (a, b).

Proof of Theorem 1 ii

If $F(x) \not\equiv 0$, then $\max F(x) > 0$ or $\min F(x) < 0$ holds. If the latter case holds, we replace $f(x)$ by $-f(x)$ to reduce this case to $\max F(x) > 0$. Then we may assume that there exists $c \in [a, b]$ such that $F(c) = \max F(x)$. It follows that $c \in (a, b)$ since $F(c) > 0 = F(a) = F(b)$. Set $\delta := \min\{c - a, b - c\}$. Since $F(c)$ is the maximum value of $F(x)$, we have for any $0 < h \leqq \delta_0$

$$\frac{F(c+h) - F(c)}{h} \leqq 0 \leqq \frac{F(c-h) - F(c)}{-h}.$$

Since $F(x)$ is differentiable at $x = c$, the both hand sides of the above converge to $F'(c)$ as $h \downarrow 0$. Then we have $F'(c) \leqq 0 \leqq F'(c)$ to imply $F'(c) = 0$. This completes the proof. □

Taylor's Theorem

We have a generalization of the mean value theorem for smoother functions.

Theorem 2

Let I be an open interval and let $a \in I$. Suppose that $f(x) \in C^N(I)$ with some $N = 1, 2, 3, \ldots$. Then we have for any $x \in I$

$$f(x) = \sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

$$R_n(x) = \frac{f^{(N)}(\theta x + (1-\theta)a)}{N!}(x-a)^N = \frac{(x-a)^N}{(N-1)!}\int_0^1 (1-t)^{N-1} f^{(N)}(tx + (1-t)a)\,dt,$$

where $\theta \in (0,1)$ is a constant depending on x, f and N. $\sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is said to be the Taylor polynomial of f at a of order $N-1$.

Proof of Theorem 2 i

Fix arbitrary $x \in I$. Suppose that $f(x) \in C^N(I)$.

Firstly we shall show the case that $R_n(x)$ is given by an integration. Then $tx + (1-t)a \in I$ for $t \in [0,1]$ and if we set $F(t) := f\big(tx + (1-t)a\big)$ then $F(t) \in C^N[0,1]$. Repeating the integration by parts, we deduce that

Proof of Theorem 2 ii

$$
\begin{aligned}
F(1) &= F(0) + \{F(1) - F(0)\} = F(0) + \int_0^1 F'(t)dt \\
&= F(0) + \Big[-(1-t)F'(t)\Big]_0^1 + \int_0^1 (1-t)F''(t)dt \\
&= \sum_{n=0}^{1} \frac{F^{(n)}(0)}{n!} + \int_0^1 (1-t)F''(t)dt \\
&= \sum_{n=0}^{1} \frac{F^{(n)}(0)}{n!} + \left[-\frac{(1-t)^2}{2!}F^{(2)}(t)\right]_0^1 + \frac{1}{2!}\int_0^1 (1-t)^2 F^{(3)}(t)dt \\
&= \cdots \\
&= \sum_{n=0}^{N-1} \frac{F^{(n)}(0)}{n!} + \frac{1}{(N-1)!}\int_0^1 (1-t)^{N-1} F^{(N)}(t)dt.
\end{aligned}
$$

Proof of Theorem 2 iii

If we substitute $F^{(n)}(0) = (x-a)^n f^{(n)}(a)$ with $n = 0, 1, \ldots, N-1$ and $F^{(N)}(t) = (x-a)^N f^{(N)}\big(tx + (1-t)a\big)$ into the above, we obtain Taylor's formula with $R_N(t)$ given by an integration.

Next we shall show the other expression of $R_N(x)$. Set

$$G(t) := f(x) - \sum_{n=0}^{N-1} \frac{f^{(n)}(t)}{n!}(x-t)^n - K(x-t)^N,$$

where K is a constant determined by $G(a) = 0$. Note that $G(x) = 0$ and $R_N(x) = K(x-a)^N$. Then the mean value theorem implies that there exists $\theta \in (0,1)$ such that $G'\big(\theta x + (1-\theta)a\big) = 0$.

Proof of Theorem 2 iv

We compute $G'(t)$:

$$G'(t) = -\sum_{n=0}^{N-1} \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \sum_{n=1}^{N-1} \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + NK(x-t)^{N-1}$$
$$= \left\{ -\frac{f^{(N)}(t)}{(N-1)!} + NK \right\} (x-t)^{N-1}.$$

Hence $G'\big(\theta x + (1-\theta)a\big) = 0$ implies thta

$$R_N(x) = K(x-a)^N = \frac{f^{(N)}\big(\theta x + (1-\theta)a\big)}{N!}(x-a)^N.$$

This completes the proof. □

Examples of Taylor's Theorem i

For $x \in \mathbb{R}$, there exists $\theta \in (0,1)$ such that

$$
\begin{aligned}
e^x &= \sum_{n=0}^{N-1} \frac{x^n}{n!} + \frac{e^{\theta x} x^N}{N!}, && (1)\\
\cos x &= \sum_{k=0}^{K-1} \frac{(-1)^k x^{2k}}{(2k)!} + \frac{(-1)^K \cos(\theta x) x^{2K}}{(2K)!} && (2)\\
&= \sum_{k=0}^{K-1} \frac{(-1)^k x^{2k}}{(2k)!} + \frac{(-1)^K \sin(\theta x) x^{2K-1}}{(2K-1)!}, && (3)\\
\sin x &= \sum_{k=0}^{K-1} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^K \cos(\theta x) x^{2K+1}}{(2K+1)!} && (4)\\
&= \sum_{k=0}^{K-1} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^K \sin(\theta x) x^{2K}}{(2K)!}. && (5)
\end{aligned}
$$

These are obtained by

$$\frac{d^n}{dx^n}e^x = e^x, \quad \frac{d^n}{dx^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right), \quad \frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right).$$

In particular, we used the following identities:

$$\frac{d^{2k}}{dx^{2k}}\cos x = (-1)^k \cos x, \quad \frac{d^{2k-1}}{dx^{2k-1}}\cos x = (-1)^k \sin x,$$

$$\frac{d^{2k+1}}{dx^{2k+1}}\sin x = (-1)^k \cos x, \quad \frac{d^{2k}}{dx^{2k}}\sin x = (-1)^k \sin x.$$

Examples of Taylor's Theorem iii

For $x > -1$ there exists $\theta = \theta(x, N) \in (0,1)$ such that

$$\log(1+x) = \sum_{n=1}^{N-1} \frac{(-1)^{n-1}}{n} x^n + \frac{(-1)^{N-1} x^N}{N(1+\theta x)^N}, \tag{6}$$

$$(1+x)^\alpha = \sum_{n=0}^{N-1} \binom{\alpha}{n} x^n + \binom{\alpha}{N} (1+\theta x)^{\alpha - N} x^N, \quad \alpha \in \mathbb{C}, \tag{7}$$

where

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \quad (n = 1, 2, 3, \dots), \qquad \binom{\alpha}{0} := 1.$$

If $\alpha = \pm m$, $m = 1, 2, 3, \cdots$, then

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!(m-n)!} & (n \leqq m), \\ 0 & (n > m), \end{cases} \qquad \binom{-m}{n} = (-1)^n \frac{(m+n-1)!}{n!(m-1)!}.$$

For $x \in \mathbb{R}$

$$\operatorname{Arctan} x = \sum_{k=0}^{K-1} \frac{(-1)^k}{2k+1} x^{2k+1} + (-1)^K x^{2K+1} \int_0^1 \frac{t^{2K}}{1+t^2x^2} dt. \tag{8}$$

When $x = 0$ the both hands sides are 0 and the formula holds. We shall show this for $x \neq 0$. Note that

$$\frac{d}{dx} \operatorname{Arctan} x = \frac{1}{1+x^2} = \sum_{k=0}^{K-1} (-1)^k x^{2k} + \frac{(-1)^K x^{2K}}{1+x^2}.$$

Integrate this from 0 to x and change the variable by $y = tx$. We have

$$\begin{aligned} \operatorname{Arctan} x &= \int_0^x \frac{1}{1+y^2} dy = \sum_{k=0}^{K-1} \frac{(-1)^k}{2k+1} x^{2k+1} + \int_0^x \frac{y^{2K}}{1+y^2} dy \\ &= \sum_{k=0}^{K-1} \frac{(-1)^k}{2k+1} x^{2k+1} + (-1)^K x^{2K+1} \int_0^1 \frac{t^{2K}}{1+t^2x^2} dt. \end{aligned}$$

Taylor Expansion

Suppose that $f(x) \in C^\infty(I)$, then

$$f(x) = \sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x)$$

for any $N = 1, 2, 3, \ldots$. Some smooth function $f(x)$ satisfies $R_N(x) \to 0$ as $N \to \infty$. In such cases we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

which is said to be the Taylor expansion or the Taylor series of $f(x)$ centered at $x = a$.

Examples of Taylor Series

The examples of Taylor series are the following.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}, \tag{9}$$

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad (1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad (\alpha \in \mathbb{C}), \quad x \in (-1,1), \tag{10}$$

$$\operatorname{Arctan} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad x \in [-1,1]. \tag{11}$$

Proof of (9)

We shall obtain the Taylor series of e^x. The Taylor series of $\cos x$ and $sinx$ can be obtaine in the same way. Fix arbitrary $R > 0$ and set $N_0 := \max\{l \in \mathbb{N} : l \leqq 2R\}$. Then $N_0 + 1 > 2R$. For $|x| \leqq R$ and $N > N_0$ we have

$$
\begin{aligned}
|R_N(x)| &= \frac{e^{\theta x}|x|^N}{N!} \leqq \frac{e^R R^N}{N!} = \frac{e^R R^{N_0}}{N_0!} \cdot \frac{R}{N_0+1} \cdot \cdots \cdot \frac{R}{N} \\
&\leqq \frac{e^R R^{N_0}}{N_0!} 2^{-N+N_0} = \frac{e^R (2R)^{N_0}}{N_0!} 2^{-N} \to 0 \quad (N \to \infty).
\end{aligned}
$$

This completes the proof. □

Proof of (10) i

In this case we have

$$R_N(x) = \binom{\alpha}{N} N x^N \int_0^1 \frac{(1-t)^{N-1}}{(1+tx)^{-\alpha+N\}}} dt,$$

where we can see $\log x$ as the case of $\alpha = 0$ if we regard $(-1)! = -1$. Set $m = \min\{l \in \mathbb{N} : |\alpha| \leqq l\}$. Then we have

$$\left| N \binom{\alpha}{N} \right| \leqq (m+1)! \frac{(m+N)!}{(N-1)!(m+1)!}.$$

Fix arbitrary $\rho \in (0,1)$. For $|x| \leqq \rho$ we have

$$|R_N(x)| \leqq (m+1)! \rho \frac{(m+N)!}{(N-1)!(m+1)!} \int_0^1 \frac{(\rho - t\rho)^{N-1}}{(1-t\rho)^{m+N}} dt.$$

Proof of (10) ii

We remark that $0 \leqq \rho - t\rho \leqq 1 - t\rho \leqq 1 - \rho < 1$. We need a number $s \in (\rho - t\rho, 1 - t\rho)$. Set s by

$$s := \frac{(\rho - t\rho) + (1 - t\rho)}{2} = \frac{1 + \rho - 2t\rho}{2}, \quad s - (\rho - t\rho) = \frac{1-\rho}{2} > 0.$$

Then we have

$$\begin{aligned} |R_N(x)| &\leqq (m+1)!\rho \left(\frac{2}{1-\rho}\right)^{m+1} \frac{(m+N)!}{(N-1)!(m+1)!} \\ &\quad \times \int_0^1 \left(\frac{\rho - t\rho}{1 - t\rho}\right)^{N-1} \left(\frac{(1-\rho)/2}{1 - t\rho}\right)^{m+1} dt \\ &\leqq (m+1)!\rho \left(\frac{2}{1-\rho}\right)^{m+1} \int_0^1 \left(\frac{1 + \rho - t\rho}{2(1 - t\rho)}\right)^{m+N} dt. \end{aligned}$$

Proof of (10) iii

We remark that

$$\frac{1+\rho-t\rho}{2(1-t\rho)} = 1 - \frac{1-\rho}{2(1-t\rho)}$$

is decreasing in $t \in [0,1]$. Then we have

$$0 < \frac{1+\rho-t\rho}{2(1-t\rho)} \leqq \frac{1+\rho}{2} < 1.$$

Substitute this into the integration, we obtain

$$|R_N(x)| \leqq (m+1)!\rho \left(\frac{2}{1-\rho}\right)^{m+1} \left(\frac{1+\rho}{2}\right)^{m+N} \to 0 \quad (N \to \infty).$$

This completes the proof $\square$

Proof of (11)

If $|x| \leqq 1$, then we have

$$|R_N(x)| = \left| x^{2K+1} \int_0^1 \frac{t^{2K}}{1+t^2x^2} dt \right| \leqq \int_0^1 t^{2K} dt = \frac{1}{2K+1} \to 0 \quad (K \to \infty).$$

This completes the proof. □

Animation: Convergence and Divergence of Taylor Series i

Animation: Convergence and Divergence of Taylor Series ii

A C^∞ function which cannot have its Taylor series

Theorem 3

If we define a function $f(x)$ by

$$f(x) := \exp(-1/x) \quad (x > 0), \quad f(x) := 0 \quad (x \leqq 0),$$

then $f(x) \in C^\infty(\mathbb{R})$ and $f(x)$ cannot have its Taylor series near $x = 0$.

We can prove that $f^{(n-1)}(x)$ is differentiable at $x = 0$ and $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$ by using the Taylor expansion of $e^{1/x}$ for $1/x$. This shows that $f(x) \in C^\infty(\mathbb{R})$ and for $0 < x \ll 1$

$$f(x) = e^{-1/x} > 0 = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Proof of Theorem 3 i

It is obvious that $f(x) \in C^\infty(\mathbb{R} \setminus \{0\})$. So it suffices to show that $f^{(n-1)}(x)$ is differentiable at $x = 0$ and $f^{(n)}(0) = 0$ inductively on $n = 1, 2, 3, \ldots$. We have $f(0) = 0$ by the definition of $f(x)$. Suppose that $f^{(n-1)}(0) = 0$. Since

$$f^{(n-1)}(x) - f^{(n-1)}(0) = \begin{cases} \text{a polynomial of } 1/x \text{ of order } 2(n-1) \times e^{-1/x} & (x > 0), \\ 0 & (x < 0). \end{cases}$$

Then there exists $C_n > 0$ such that for $0 < |x|$

$$|f^{(n-1)}(x) - f^{(n-1)}(0)| \leqq \frac{C_n e^{-1/|x|}}{x^{2n-2}}.$$

The Taylor series of the exponential function implies that for $x \neq 0$

$$0 < e^{-1/|x|} = \frac{1}{e^{1/|x|}} = \frac{1}{\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!|x|^k}} \leqq \frac{1}{\frac{1}{(2n)!x^{2n}}} = (2n)!x^{2n}.$$

Then we have for $0 < |x| \leqq 1$

$$\left| \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x} - 0 \right| \leqq \frac{|f^{(n-1)}(x) - f^{(n-1)}(0)|}{|x|}$$

$$\leqq \frac{C_n e^{-1/|x|}}{|x|^{2n-1}} \leqq C_n (2n)! |x| \to 0 \quad (x \to 0),$$

which shows that $f^{(n-1)}(x)$ is differentiable at $x = 0$ and $f^{(n)}(0) = 0$. This completes the proof. □